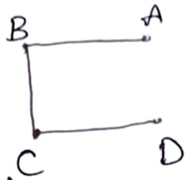
Graph Theory

Definition:-

Graph :- A graph is a pair (V, E) the elements of V are called vertices and all elements of E are called undirected edges or edges. denoted by G = (V, E)



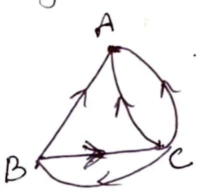
V = {A, B, C D}

E = {AB, BC, CD}

Directed graph (Digraph)

A directed graph (digraph) is a pair (V, E) where V is a nonempty set and E is a set of ordered pair of elements taken from the set V.

Digraph (V, E) the elements of V are called vertices and the elements of E are called directed edges. The set V is called the vertex set & the set E is called the directed edge set. denoted by D = (V, E)



V = {A, B, C}

E = {BA, CA, CA, BC, CB}

If BA is a directed edges of a digraph D the we say B is the initial vertex & A is the terminal vertex

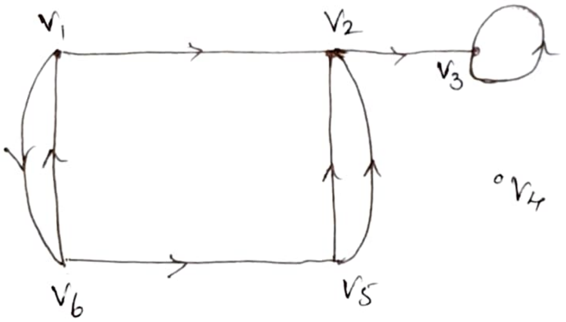
A directed edge beginning and ending at the same vertex. A is denoted by AA and is called a directed loop.

Two directed edges having the same initial vertex and the same terminal vertex are called parallel directed edges.

Two or more directed edges having the same initial vertex & the same terminal vertex are called multiple directed edges.

Indegree & Out degree.

If v is a vertex of a digraph D. the number of edges for which v is the initial vertex is called the out going degree or the out degree of v and the number of edges for which v is the terminal vertex is called the incoming degree or the in degree. The out degree of v is denoted by $d^+(v)$ or $od(v)$ and the in degree of v is denoted by $d^-(v)$ or $id(v)$



$d^+(v_1) = 2$	$d^-(v_1) = 1$
$d^+(v_2) = 1$	$d^-(v_2) = 3$
$d^+(v_3) = 1$	$d^-(v_3) = 1$
$d^+(v_4) = 0$	$d^-(v_4) = 0$
$d^+(v_5) = 2$	$d^-(v_5) = 1$
$d^+(v_6) = 2$	$d^-(v_6) = 1$

Null graph

A graph containing no edges is called null graph.

Eg:- A, B, C

Trivial graph

A null graph with only one vertex is called a trivial graph. • A

~~c graph~~ Finite graph

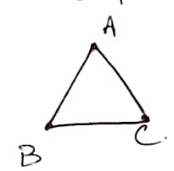
A graph with only a finite number of vertices as well as only a finite number of edges is called a finite graph otherwise it is called infinite graphs.

Order & Size

The number of vertices in a (finite) graph is called the order of the graph & the number of edges in it is called its size. For a Graph $G=(V,E)$ the cardinality of the set V namely $|V|$ is called the order of G & the cardinality of the set E namely $|E|$ is called the size of G. It is denoted by (n,m)

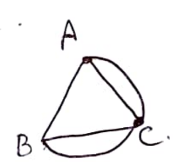
Simple Graph

A graph which does not contain loops and multiple edges. is called a simple graph.



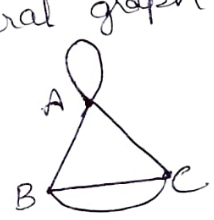
Multigraph

A graph which contains multiple edges but no loops is called a multigraph.



General graph.

A graph which contains multiple edges or loops or both is called a general graph.



~~Incidence~~ Incidence

~~When~~ When a vertex v of a graph G is an end vertex of an edge e of the graph G. Then we say that the edge e is incident on the vertex v.

Two non parallel edges are said to be adjacent edges if they are incident on a common vertex. Two vertices are said to be adjacent vertices if ~~there~~ there is an edge joining them.

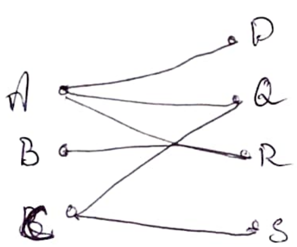
Complete Graph.

A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a complete graph. denoted by K_n.

K_5 is called the Kuratowski's first graph.

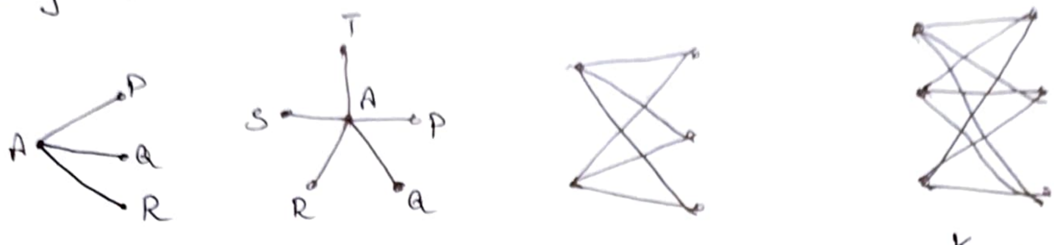
Biparbite Graph

A simple graph G is such that its vertex set V is the union of two mutually disjoint nonempty set V_1 & V_2 which are such that each edges in G joins a vertex in V_1 and a vertex in V_2. Then G is called a biparite graph. denoted by $G = (V_1, V_2; E)$



plete Bipartite Graph Complete Bipartite Graph

A bipartite graph $G = (V_1, V_2 ; E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

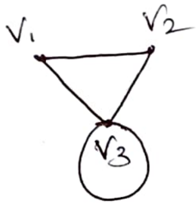


$K_{1,3}$ $K_{1,5}$ $K_{2,3}$ $K_{3,3}$.

The Graph $K_{3.3}$ is known as the Kuratowski's second graph.

Vertex degree

The number of edges of G that are incident on v (ie, the number of edges that join v to other vertices of G) with the loops counted twice is called degree of the vertex v

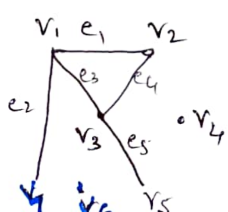


$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 4$

Isolated vertex. Pendant vertex

A vertex in a graph which is not an end vertex of any edge of the graph is called an isolated vertex. a vertex is an isolated vertex if & only if its degree is zero.

A vertex of degree 1 is called a pendant vertex. An edge incident on a pendant vertex is called a pendant edge



v_4 & v_8 are isolated vertex.

v_5 & v_7 are pendant vertex

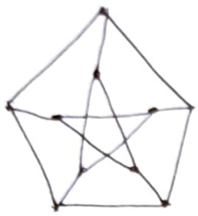
e_5 & e_2 are pendant edge.

egular graph Regular graph

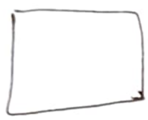
A graph in which all the vertices are of the same degree is called a regular graph.

A regular graph in which all vertices are of degree K is called K-regular graph.

3 regular graph are called cubic graph.



→ 3 regular graph with 10 vertices & 15 edges. is called the petersen graph.



→ 2 regular.

- 4-regular

Handshaking property

The sum of the degrees of all the vertices in a graph is an even number and this number is equal to twice the number of edges in the graph.

$$\sum_{v \in V} \deg(v) = 2|E|$$

Theorem In every graph the number of vertices of odd degrees is even.

Proof :- Consider a graph with n vertices. Suppose K of these vertices are of odd degree so that the remaining n-k vertices are of even degree. The odd degree vertices are $v_1\ v_2\ v_3 \ldots v_K$ and the vertices with even degree by $v_{K+1}, v_{K+2} \ldots v_n$. The sum of the degrees of the vertices is $\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{K} \deg(v_i) + \sum_{i=K+1}^{n} \deg(v_i)$

∴ Using hand shaking property. The sum on the left hand side of the above expression is equal to twice the number of edges in the graph. As such this sum is even

Further the second sum in the right hand side is the sum of the degrees of vertices with even degrees.

As such this sum is also even

∴ the first sum in the right hand side must also be even that is

$$deg(v_1) + deg(v_2) + \dots + deg(v_k) = \text{even}.$$

But each of $deg(v_1)$ $deg(v_2)$ -- $deg(v_k)$ is odd ∴ The no of terms in the left hand side of 2 must be even

that is k is even.

Problems.

1. For a graph with n vertices and m edges. if δ is the minimum and Δ is the maximum of the degrees of vertices. show that $\delta \leq \frac{2m}{n} \leq \Delta$

Soln: Let $d_1, d_2, d_3 \dots d_n$ be the degrees of the vertices

Then by handshaking property.

$$d_1 + d_2 + d_3 + \dots + d_n = 2m \quad \text{——} \quad (i)$$

Since $\delta = \min(d_1, d_2\, d_3 \dots d_n)$ we have

$$d_1 \geq \delta \quad d_2 \geq \delta \dots \quad d_n \geq \delta$$

Adding these n inequalities we get

$$d_1 + d_2 + \dots + d_n \geq n\delta \quad \text{——} \quad (ii)$$

III^{ly} Since $\Delta = \max(d_1, d_2 \dots d_n)$ we get

$$d_1 + d_2 + \dots + d_n \leq n\Delta \quad \text{——} \quad (iii).$$

(i), (ii) & (iii) we get $2m \geq n\delta$ & $2m \leq n\Delta$

$$n\delta \leq 2m \leq n\Delta$$

$$\delta \leq \frac{2m}{n} \leq \Delta$$

(2) Let D be a directed graph with n vertices. If the underlying graph of D is K_n. prove that

$$\sum_{v \in V} [od(v)]^2 = \sum_{v \in V} [id(v)]^2$$

Soln

$\sum_{v \in V} od(v) = \sum_{v \in V} id(v)$ [The sum of the outdegree of all vertices is equal to the sum of the indegrees of all vertices]

$$\sum_{v \in V} od(v) - id(v) = 0 \quad \text{— (1)}$$

$$d(v) = id(v) + od(v) \quad \forall\ v \in V$$

If the underlying graph is K_n we have $d = n-1\ \forall\ v \in V$

$\therefore$ K_n is the underlying graph of the given digraph D

we have $id(v) + od(v) = n-1 \quad \forall\ v \in D$ — (2).

$$\sum_{v \in V} [\{od(v)\}^2 - \{id(v)\}^2] = \sum_{v \in V} \{od(v) + id(v)\}\{od(v) - id(v)\}$$

$= (n-1) \sum \{od(v) - id(v)\}$ using (2)

$= (n-1)(0)$ using (1)

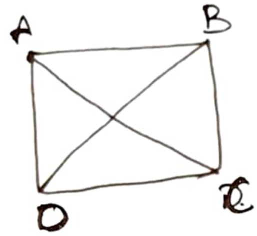
$= 0$

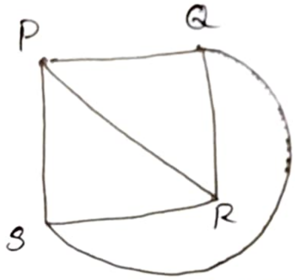
This gives $\sum_{v \in V} [od(v)]^2 = \sum_{v \in V} [id(v)]^2$ as required.

Definition: Two Graphs G and G' are said to be isomorphic if there is a one to one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved.

G and G' are isomorphic we write $G \cong G'$

when a vertex A of G corresponds to the vertex $A' = f(A)$ of G' under an one to one correspondence $f: G \to G'$ we write $A \leftrightarrow A'$. III^ly we $\{A, B\} \leftrightarrow \{A', B'\}$ to mean that the edge AB of G & the edge $A'B'$ of G' correspond to each other.





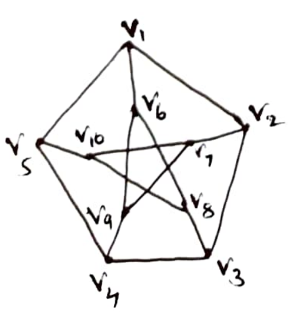
$A \leftrightarrow P$ $B \leftrightarrow Q$ $D \leftrightarrow S$ $C \leftrightarrow R$.

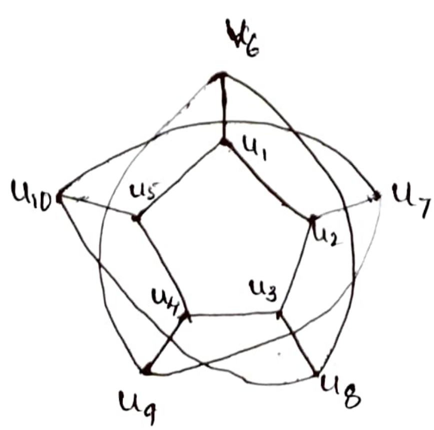
$AB \leftrightarrow PQ$ $AC \leftrightarrow PR$ $AD \leftrightarrow PS$ $BC \leftrightarrow QR$ $BD \leftrightarrow QS$

$CD \leftrightarrow RS$

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.

Problems





$v_i \leftrightarrow u_i$ for $i = 1, 2, 3, \ldots 10$.

① one to one correspondence between the edges in two graphs

② adjacent vertices in the first graph correspond to the adjacent vertices in the second graph. & vice-versa.

∴ Two graphs are isomorphic.

$v_1 \leftrightarrow u_1$ $v_2 \leftrightarrow u_2$ $v_3 \leftrightarrow u_3$ $v_4 \leftrightarrow u_4$ $v_5 \leftrightarrow u_5$ $v_6 \leftrightarrow u_6$ $v_7 \leftrightarrow u_7$

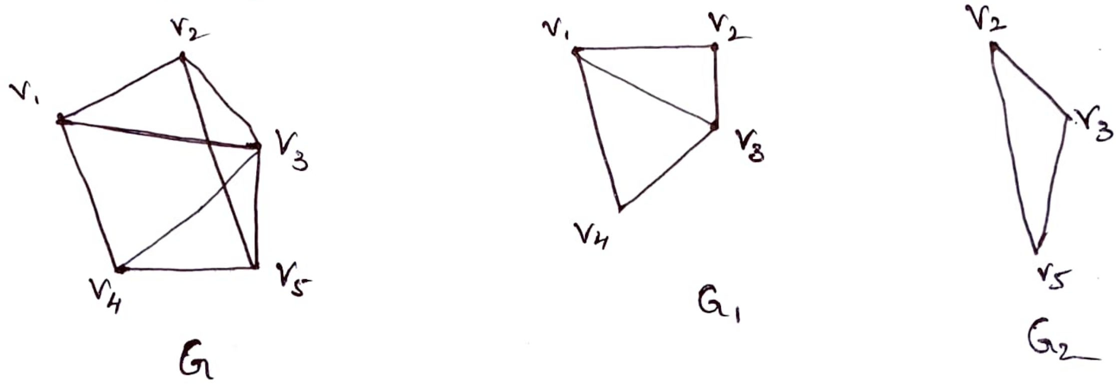
$v_8 \leftrightarrow u_8$ $v_9 \leftarrow u_9$ $v_{10} \leftrightarrow u_{10}$.

Subgraphs

Given two graphs G and G_1. ~~then~~ G_1 is a subgraph of G. if the following conditions hold.

① All the vertices and all the edges of G_1 are in G.

② Each edge of G_1 has the same end vertices in G as in G_1



The following results are immediate consequences of the definition of a subgraph.

① Every graph is a subgraph of itself

② Every simple graph of n vertices is a subgraph of the complete graph K_n

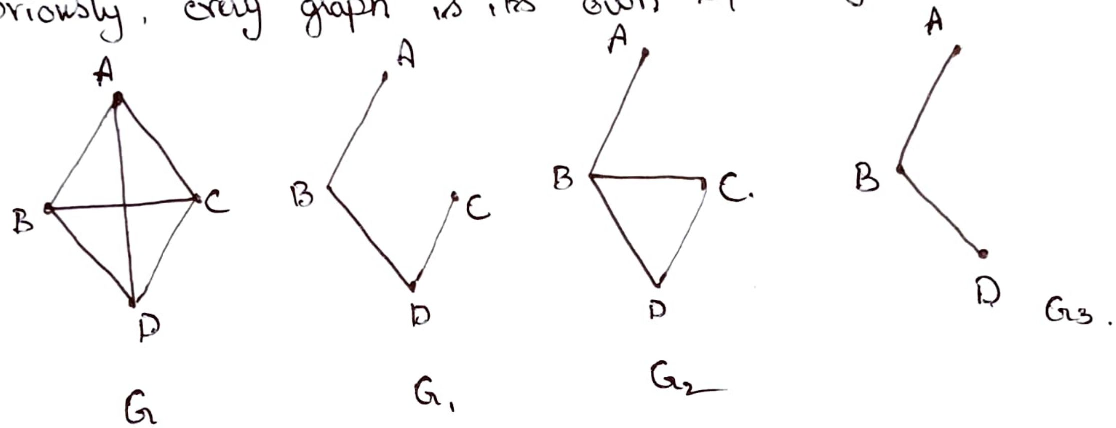
③ If G_1 is a subgraph of a graph G_2 & G_2 is a subgraph of a graph G. then G_1 is a subgraph of G.

4) A single vertex in a graph G is a subgraph of G

(5) A single edge in a graph G together with its end vertices is a subgraph of G.

Spanning Subgraph

A subgraph G_1 of a graph G is a spanning subgraph of G whenever the vertex set G_1 contains all vertices of G. Thus a graph and all its spanning subgraphs have the same vertex set. Obviously, every graph is its own spanning subgraph.



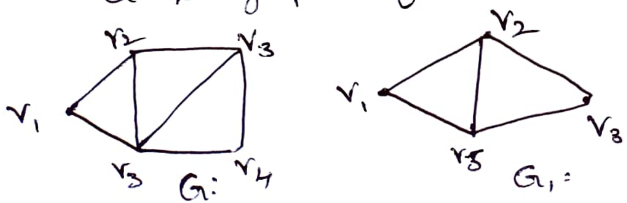
G_1 & G_2 are spanning subgraph of G.

G_3 is not a spanning subgraph of G.

[A graph $G = (V, E)$ if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$ then G_1 is called a spanning subgraph of G]

Induced subgraph

Given a graph $G = (V, E)$ suppose there is a subgraph $G_1 = (V_1, E_1)$ of G such that every edge $\{A, B\}$ of G where $A, B \in V_1$ is an edge of G_1 also. Then G_1 is called a subgraph of G induced by V_1 & is denoted by $\langle V_1 \rangle$



$V_1 = \{v_1, v_2, v_3, v_5\}$

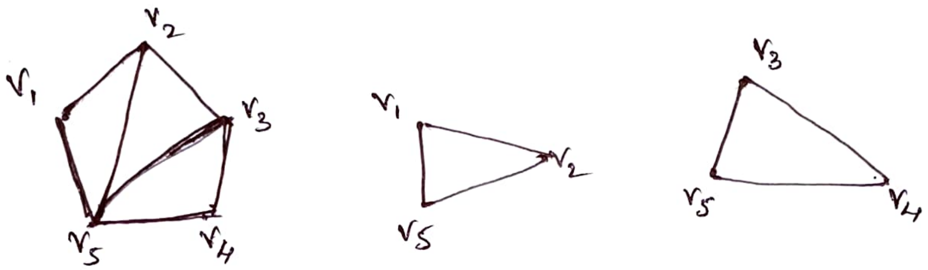
Edge disjoint and vertex disjoint Subgraphs.

Let G be a graph and G_1 & G_2 be two subgraphs of G. Then

1. G_1 & G_2 are said to be edge disjoint if they do not have any common edge.
2. G_1 & G_2 are said to be vertex disjoint if they do not have any common edge and any common vertex.

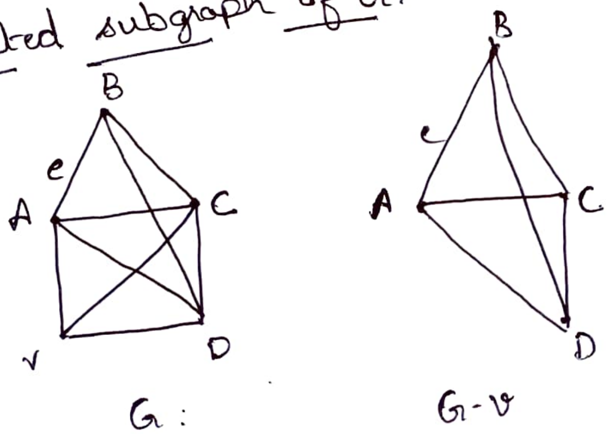
Note:- Edge-disjoint subgraphs may have common vertices. Subgraphs that have no vertices in common cannot possibly have edge in common.

2. Two vertex-disjoint subgraphs must be edge-disjoint but the converse is not necessarily true.



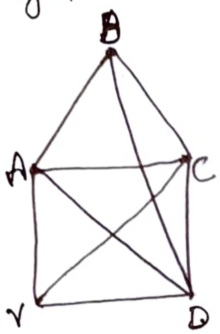
Deletion

If v is a vertex in a graph G then G-v denotes the subgraph of G obtained by deleting v and all edges incident on v from G. This subgraph G-v is referred to as a vertex deleted subgraph of G.

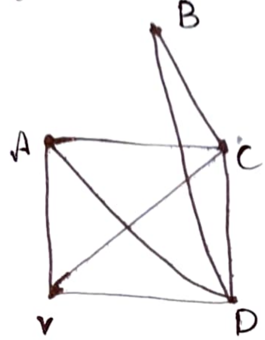


Note:- Deletion of a vertex always results in the deletion of all edges incident on that vertex.

If e is an edge in a graph G, then G-e denotes the subgraph of G obtained by deleting e from G. This subgraph G-e is referred to as edge deleted subgraph of G



G

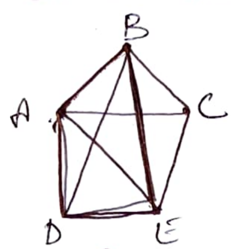


G-e

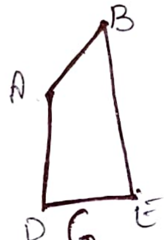
Note:- Deletion of an edge does not alter the number of vertices. An edge deleted subgraph of a graph is a spanning subgraph of the graph.

Complement of a subgraph:-

A Graph G and a subgraph G_1 of G, the subgraph of G obtained by deleting from G all the edges that belong to G_1 it is denoted by $G - G_1$ or $\overline{G_1}$ is called Complement of G_1 in G



G



G_1

$G_1' = G - G_1$

Complement of a Graph.

Every simple graph of order n is a subgraph of the complete graph K_n. If G is a simple graph of order n the complement of G is K_n is called the complement of G it is denoted by $\overline{G}$

The complement $\overline{G}$ of a simple graph G with n vertices is that graph which is obtained by deleting those edges in K_n which belong to G. $\overline{G} = K_n - G$ or $K_n \Delta G$

Note $\overline{\overline{G}} = G$ (2) Complement of K_n is the null graph of order n & vice versa.

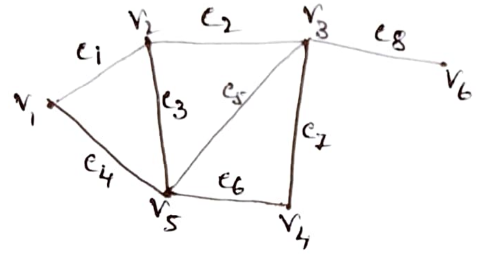
K_4 $\quad$ G. $\quad$ $\bar{G} = K_4 - G$

K_4 $\quad$ $\bar{G} = K_4 - G$

Walks and their Classification.

G consider a finite alternating sequence of vertices and edges of the form

$v_i\, e_j\, v_{i+1}\, e_{j+1}\, v_{i+2} \cdots e_k v_m$ which begins & ends with vertices & which is such that each edge in the sequence is incident on the vertices preceding.



The number of edges present in a walk is called its length.

Vertex with which a walk begins is called the initial vertex. (origin). The vertex with which a walk ends is called the final vertex (or terminus)

(initial vertex & final vertex of a walk are together called its terminal vertices).

A walk that begins & ends at the same vertex is called a closed walk.

A walk that is not closed is called an open walk.

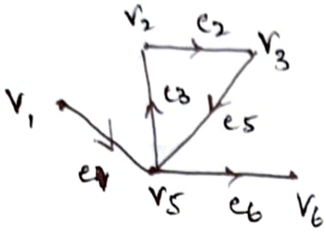
An open walk is a walk that begins & ends at two different vertices.

(i) The sequence $v_1 e_1 v_2 e_2 v_3 e_8 v_6$ is walk of length 3

(ii) The sequence $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ is walk of length 5.

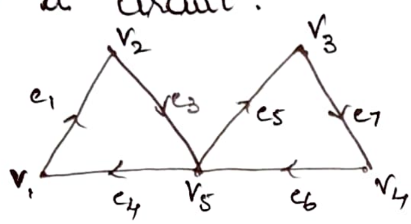
Trail & Circuit

If in an open walk no edges appears more than once, then the walk is called a trail.



$v_1 e_1 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_6$.

A closed walk in which no edge appears more than once is called a circuit.



$v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4 e_6 v_5 e_4 v_1$

Path and Cycle

A trail in which no vertex appears more than once is called a path.

A circuit in which the terminal vertex does not appear as an internal vertex & no internal vertex is repeated is called a cycle.

The following facts are to be emphasised.

1. A walk can be open or closed. In a walk a vertex and an edge can appear more than once.
2. A trail is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.
3. A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.
4. A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail but a trail need not be a path.

A cycle is a closed walk in which neither a vertex nor an edge can appear more than once. Every cycle is a circuit need not be a cycle.

6. If a cycle contains only one edge it has to be a loop.

7. Two parallel edges (when they occur) form a cycle.

8. In a simple graph, a cycle must have at least 3 edges. (A cycle formed by 3 edges is called a triangle).

Connected & Disconnected Graphs.

Two vertices in G are said to be connected if there is atleast one path from one vertex to the other.

G is connected if every pair of distinct vertices in G are connected otherwise G is called a disconnected graph.

Connected.

disconnected.

A Graph G is said to be.

(i) Connected if there is at least one path between every two distinct vertices in G and.

(ii) disconnected if G has atleast one pair of distinct vertices between which there is no path.

Components:-

Theorem 1:- If a graph has exactly two vertices of odd degree then there must be a path connecting these vertices.

*

Theorem 2:- A simple graph with n vertices and k components, can have atmost $(n-k)(n-k+1)/2$ edges.

Proof:- Let n_1 be the no of vertices in 1^{st} component.

n_2 be the no of vertices in 2^{nd} ~~component~~ component.

n_k be the no of vertices in k^{th} component

Then $n_1 + n_2 + \dots + n_k = n.$ — (i)

Then gives

$$(n_1-1) + (n_2-1) + \dots + (n_k-1) = n - (1+1+\dots + k \text{ terms})$$

$$= n-k$$

Squaring both sides.

$$(n_1-1)^2 + (n_2-1)^2 + \dots (n_k-1)^2 + s = (n-k)^2 \rightarrow \text{(ii)}$$

where S is the sum of products of the form $2(n_i-1)(n_j-1)$

$i = 1, 2 \dots k \quad j = 1, 2 \dots k \quad i \neq j$

Since each of $n_1\, n_2 \dots n_k$ is greater than or equal to 1

we have $S \geq 0$

$\therefore$ eqn (ii)

$$(n_1-1)^2 + (n_2-1)^2 + \dots + (n_k-1)^2 \leq (n-k)^2$$

or $n_1^2 + n_2^2 + \dots + n_k^2 - 2(n_1 + n_2 + n_3 + \dots + n_k) + k \leq (n-k)^2$

$$n_1^2 + n_2^2 + \dots + n_k^2 \leq (n-k)^2 + 2n - k \quad \text{using (i)}$$

$$= n^2 + k^2 - 2nk + 2n - k$$

$$= n^2 - (k-1)(2n-k)$$

ie $\displaystyle\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$ — (iii)

Since G is a simple graph each of the components of G. is a simple graph.

$\therefore$ The maximum no. of edges which the ith component can have $\frac{1}{2} n_i(n_i-1)$

$\therefore$ the maximum no. of edges which G can have is N (Complete Graph)

$$N = \frac{1}{2} \sum_{i=1}^{k} n_i(n_i-1) \quad \text{— (iv)}$$

From (iv)

$$N = \frac{1}{2} \sum_{i=1}^{k} (n_i^2 - n_i) = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \quad [\text{using (i)}]$$

$$\leq \frac{1}{2} [n^2 - (k-1)(2n-k)] - \frac{1}{2} n \quad \text{using (iii)}$$

$= \frac{1}{2}(\underbrace{n^2 - 2nk + n + k^2}_{} - k) = \frac{1}{2}\{(n-k)^2 + (n-k)\}$

$= \frac{1}{2}(n-k)(n-k+1) \qquad = \frac{1}{2}(n-k)\{n-k+1\}$

The number of edges in G cannot exceed $\frac{1}{2}(n-k)(n-k+1)$

Theorem 3 :- A connected graph with n vertices has atleast (n-1) edge.

Solu Since the graph is connected $n \geq 2$. If m denotes the number of edges.

To prove that $m \geq n-1$ for every positive integer $n \geq 2$

By the method of induction to prove this results.

Suppose $n = 2$. Then there are two vertices in the graph & since the graph is connected. there must be at least one edge joining these vertices.

$m \geq 1 = (2-1) = (n-1)$ This verifies the required result for $n = 2$.

Assume that result $m \geq n-1$ holds for all connected graphs with $n = K$ no of vertices where K is a positive integer ≥ 2

Now consider a connected graph say G_{K+1} with K+1 vertices. choose a vertex v of this graph and consider the graph G_K obtained by deleting an edge from G_{K+1} for which v is an end vertex. Then G_K is a connected graph with K vertices. Let m_K be the no of edges in G_K.

Then from the assumption made in the preceding paragraph we have $m_K \geq K-1$

Consequently $m_K + 1 \geq (K+1) - 1$

But $m_K + 1$ is the no of edges in G_{K+1} & K+1 is no of vertices in G_{K+1}. Thus the result $m \geq n-1$ holds for $n = K+1$ when it holds for $n = k \geq 2$.

Hence by induction the result holds for all integers $n \geq 2$

Theorem 4:- A graph G is disconnected if and only if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other is in V_2.

Euler Circuits & Euler trails.

A Connected graph G. If there is a circuit in G that contains all the edges of G. then that circuit is called an Euler circuit (or Eulerian line, or Euler tour) in G.

If there is a trail in G that contains all the edges of G, then that trail is called an Euler trail in G.

[In a trail & a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails & Euler circuits also].

Since Euler Circuits & Euler trails include all the edges. They automatically should include all vertices as well.

→ A Connected graph that contains an Euler circuit is called an Euler graph or Eulerian graph.

→ A Connected graph that contains an Euler trail is called a semi Euler graph (or semi Eulerian graph or unicursal graph)

Theorem 1:- A connected graph G has an Euler circuit if and only if all vertices of G are of even degree.

Theorem 2 :- A Connected graph G has an Euler circuit if and only if G can be decomposed into edge-disjoint Cycles.

Hamilton Cycles & Hamilton paths.

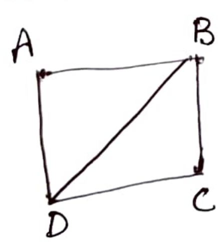
Let G be a connected graph. If there is a cycle in G that contains all the vertices of G. then that cycle is called a Hamilton cycle in G.

→ A Hamilton cycle in a graph of n vertices consists of exactly n edges Because a cycle with n vertices has n edges.

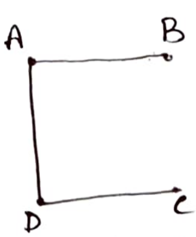
→ A Hamilton cycle in a graph G must include all vertices in G. This does not mean that it should include all edges of G.

→ A graph that contains a Hamilton cycle is called a Hamilton graph or Hamiltonian graph.

→ A path in a connected graph which includes every vertex of the graph is called Hamilton / Hamiltonian path in the graph.



(ABCD is hamilton cycle)



(CDAB is hamilton path)

→ Hamilton path in a graph G meets every vertex of G, the length of a hamilton path in a connected graph of n vertices is n-1

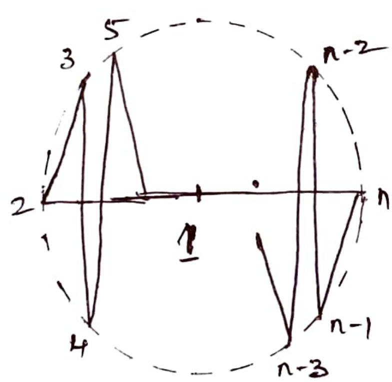
Theorem 1 :- A simple connected graph with n vertices (where $n \geq 3$) is Hamiltonian if the sum of the degrees of every pair of non adjacent vertices is greater than or equal to n.

Theorem 2 :- A simple connected graph with n vertices (where $n \geq 3$) is Hamiltonian if the degree of every vertex is greater than or equal to $n/2$

Theorem 3 :- In the complete graph with n vertices where n is an odd number ≥ 3 there are $(n-1)/2$ edge disjoint Hamiltonian cycles.

Proof :- Let G be a complete graph with n vertices, where n is odd & ≥ 3.

Denote the vertices of G by $1, 2, 3 \ldots n$ and represent them as points as shown in figure.



We note that the polygonal pattern of edges from vertex 1 to vertex n as depicted in the figure is a cycle that includes all the vertices of G. This cycle is therefore a Hamilton Cycle. This representation demonstrates that G has atleast one Hamilton Cycle.

Now rotate the polygonal pattern clockwise by $\alpha_1, \alpha_2 \ldots \alpha_K$ degrees where $\alpha_1 = \frac{360}{n-1}$ $\alpha_2 = 2\,\frac{360}{n-1}$

$\alpha_3 = 3\,\frac{360}{n-1}$ .... $\alpha_K = \frac{n-3}{2}\,\frac{360}{n-1}$

Each of these $K = (n-3)/2$ rotations gives a Hamilton Cycle that has no edge in common with any of the preceding ones. Thus $\exists$ $K = (n-3)/2$ new hamilton cycles. all edges disjoint from the one shown in figure and also edge disjoint among themselves. Thus in G there are exactly

$$1 + K = 1 + \frac{n-3}{2} = \frac{1}{2}(n-1)$$ mutually edge disjoint

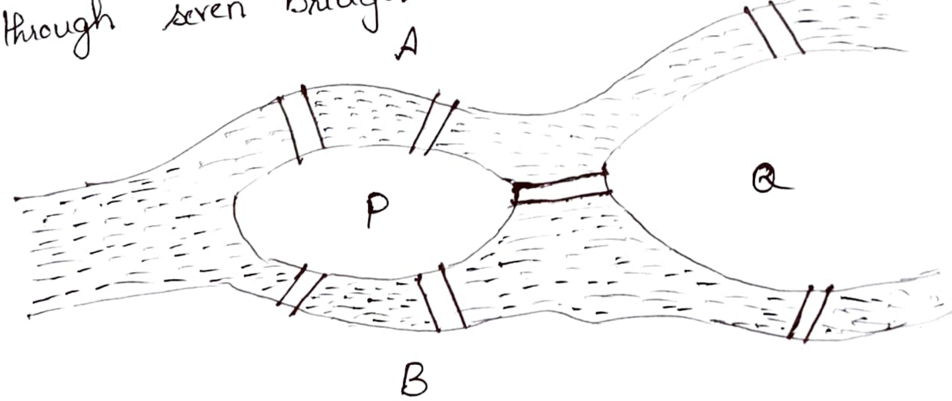
Hamilton cycles.

Some Classical Problems

Konigsberg Bridge Problem.

Konigsberg in East Prussia (Europe)

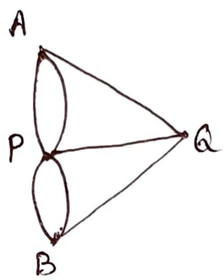
In 18 century city named Konigsberg in East Prussia (Europe) there flowed a river named Pregel River which divided the city into four parts. Two of these parts were the banks of the river & two were islands. These parts were connected with each other through seven bridges as depicted



The citizens of the city seemed to have posed the following problem. By starting at any of the four land areas, can we return to that area after crossing each of the seven bridges exactly once ?

~~In the year~~ This problem remained unsolved for several years.

In the year ~~1813~~ 1736 Euler analyzed the problem with the help of a graph and gave the solution. He denote the land areas of the city by A, B, P, Q where A, B are the banks of the river and P, Q are the islands. Construct a graph by treating the four land areas as four vertices and the seven bridges connecting them as seven edges. The graph is as shown in figure.



we note that $deg(A) = deg(B) = deg(Q) = 3$, $deg(P) = 5$ which are not even. Therefore the graph does not have an Euler circuit. This means that there does not exist a closed walk that contains all the edges exactly one. This amounts to saying that it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

Thus the solution to the Konigsberg problem is in the negative.

The Cubic graph $2^3 = 8$ vertices is Called 3 dimensional hypercube.

The Cubic graph with 2^k vertices is called k-dimensional hypercube & is denoted by Q_K

$\deg(A) = 6 \quad \deg(C) = 4$
$\deg(B) = 2$

$\deg(A) + \deg(B) + \deg(C) = 6+4+2 = 12$
$= 2 \times 6 = 2m$

Show that every simple graph of order ≥ 2 must have atleast two vertices of the same degree

Soln Let G be a simple graph with n vertices

Suppose all the vertices have different degrees.

Then since every vertex must have a degree

& since all such degrees must be between 0 & n-1

ie degree must 0, 1, 2, 3 .... n-1

Let A be the vertex whose degree is 0 & B be the vertex whose degree is n-1. Then n-1 edges are incident on B.

This means that B is joined to all other vertices by an edge and in particular to A also.

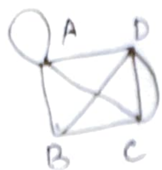
Hence the degree of A is not zero.

This is contradiction.

Hence all vertices of G cannot have different degrees.

at least two of them must have the same degree

Handshaking Property.



$d(A) = 5 \quad d(B) = 3 \quad d(C) = 4 \quad d(D) = 4$

$d(A) + d(B) + d(C) + d(D) = 16$

$= 2 \times 8 = 2m$

The sum of degrees of vertices in a graph G is equal to twice the number of edges in G.

This is because of the fact that while counting the degrees of vertices, each edge is counted twice which is same as two hands are involved in each handshake.

ie If a graph has n number of vertices then

$\deg(v_1) + \deg(v_2) + \deg v_3 + \dots + \deg(v_n) = 2m$

$$\sum_{i=1}^{n} \deg(v_i) = 2m$$

P.T The number of odd deg vertices in a Graph G is even.

Proof :- Let G be a graph with n vertices. There are K vertices of odd degree & $(n-k)$ vertices of even deg

By handshaking property

$\{\deg(v_1) + \deg(v_2) + \dots + \deg(v_k)\} +$

$\{\deg(v_{k+1}) + \deg(v_{k+2}) + \dots + \deg(v_k)\} = 2m$

$$\sum_{i=1}^{K} \deg(v_i) + \text{even} = \text{even}.$$

$$\Rightarrow \sum_{i=1}^{K} \deg v_i = \text{even}$$

The sum of degrees of vertices of odd degree = even

$\therefore$ even no of odd deg vertices.

1) If $G = (V, E)$ is a simple Graph P.T

$$2|E| \leq |V|^2 - |V|$$

or $2m \leq n^2 - n$.

Proof :- Let $G = (V, E)$ be a simple Graph with n vertices & m edges.

Graph has no loops & multiple edges. (simple Graph)

Each edge of a graph is determined by a pair of vertices. In a simple graph the number of edges cannot exceed the number of pairs of vertices. that can be chosen from n vertices in nC_2

$$nC_2 = \frac{1}{2} n(n-1)$$

In a simple graph the no of edges cannot exceed

$$nC_2 = m \leq \frac{n(n-1)}{2}$$

$$2m \leq n^2 - n.$$

$$2|E| \leq |V|^2 - |V|$$

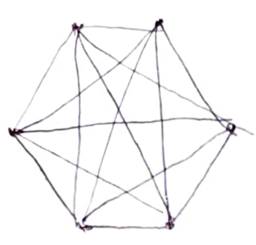
2) A Complete Graph K_n with n vertices has $\frac{1}{2}n(n-1)$ edges.

Solu: A Complete Graph is a simple graph in which there is an edge between every pair of vertices.

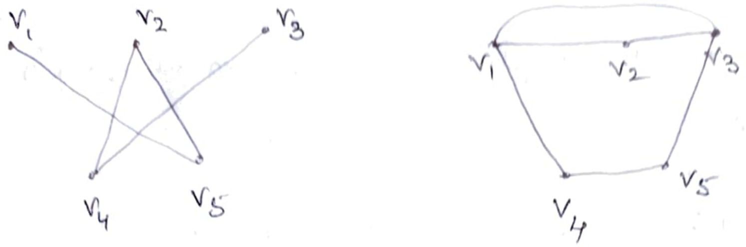
$$m = nC_2$$

$$m = \frac{1}{2} n(n-1)$$

3) Complete graph K_5



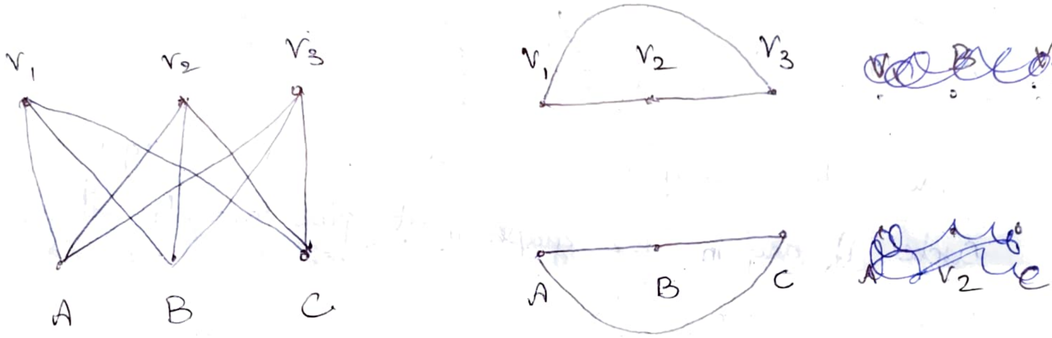
Show that the complement of a bipartite graph need not be a bipartite graph.



bipartite graph which is of order 5

The Complement of this graph is not a bipartite graph.

Find the Complement of the Complete bipartite graph $K_{3,3}$



Show that a Connected graph with exactly two vertices of odd degree has an Euler trail.

Soln: let A & B be the only two vertices of odd degree in a connected graph G.

Join these vertices by an edge e (even if there is already an edge between them). Then A and B become vertices of even degree. Since all other vertices in G are of even degree the graph $G_1 = G \cup e$ is connected and has all vertices of even degree.

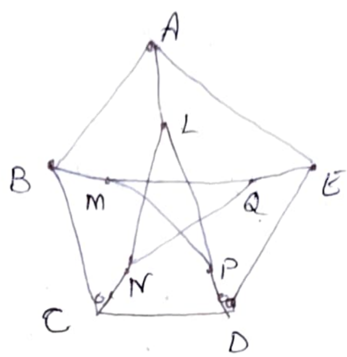
$\therefore$ G_1 contains a Euler Circuit which must include e

The trail got by deleting e from this Euler Circuit is an Euler trail in G.

Show that the peterson graph has no Hamilton cy[cle] in it but that it has a Hamilton path.

Solu: The peterson graph is a 3 regular graph with 10 verti[ces] & 15 edges

The graph is shown below with the vertices label as A B C D E L M N P Q



since the graph has 10 vertices & 15 edges a Hamilton cycle if any in the graph must pass through all the 10 vertices & must have 15 edges.

we observe that 3 edges are incident at every vertex of the graph.

of these three edges only two can be included in a Hamilton cycle (if it exists). Thus at each of the 10 vertices of the graph one edge has to ~~be~~ be excluded. By actual counting we find that the number of edges to be excluded is six (AL BM CN DE PM QE)

Consequently the no of edges that remain in the graph is 15 - 6 = 9. These edges are insufficient to form a Hamilton cycle in the graph.

Thus the peterson graph does not contain a Hamilton cycle in it.

we note that the edges AB, BC, CD, DE, EQ, QM, MP, PL, LN form a path & this path includes all the vertices. This path is therefore a Hamilton path.

Thus the peterson graph does have a Hamilton path in it.

Show that every cubic graph has an even number of vertices.

Solu: G be a cubic graph with n vertices

ie every vertex of degree 3

ie every vertex is of odd degree 3

In a Graph G the number of odd degree terms are even.

$\therefore$ Cubic graph has an even number of vertices.

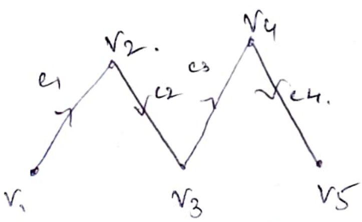
Hypercube.

<u>Component</u>

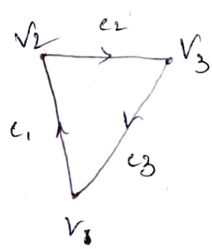
Graph G consist of one or more connected graphs. Each such connected graph is a subgraph of G and is called a component of G.

A connected graph has only one component & a disconnected graph has two or more components.

The number of components of a graph G is denoted by $K(G)$.

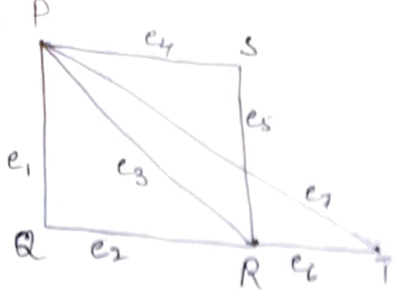


Path



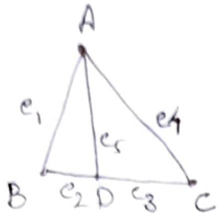
cycle

eg:-



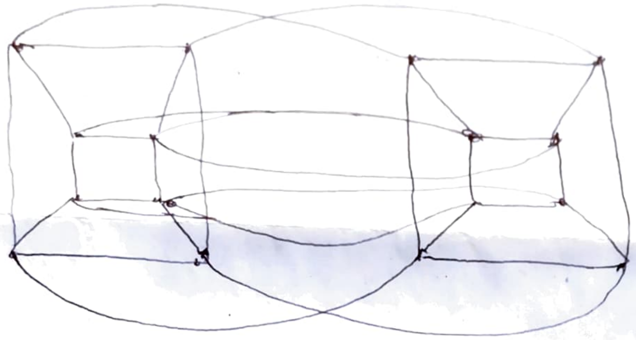
$Pe_1Qe_2Re_3Pe_4Se_5Re_6Te_7P$

Euler circuit.



$Ae_1Be_2De_3Ce_4Ae_5D$ Euler trail

Draw a diagram for the four dimensional hypercube Q_4



Prove that the K dimensional hypercube Q_K has $K2^{K-1}$ edges.

In the hypercube Q_K the number of vertices is 2^K & each vertex is of degree K.

Therefore, the sum of degrees of vertices of Q_K is $K \times 2^K$

By handshaking property

$K \times 2^K = 2|E|$ where $|E|$ is the size of Q_K.

Thus $|E| = \frac{1}{2}(K \times 2^K) = K \cdot 2^{K-1}$

$\Rightarrow$ Q_K has $K2^{K-1}$ edges.

$\therefore Q_8$ has $8 \times 2^7 = 1024$

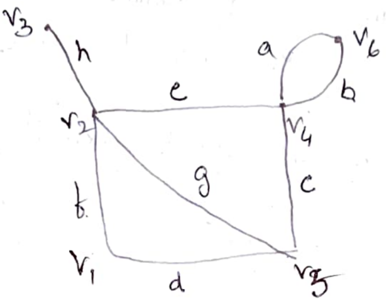
Q_3 has $3 \times 2^2 = 3 \times 4 = 12$

Incidence Matrix

G be a graph with n vertices, e edges & no self loops. Define an n by e matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and the e columns corresponds to the e edges as follows.

The matrix element

$$a_{ij} = 1 \quad \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v_i \text{ \&}$$
$$= 0 \quad \text{otherwise.}$$



$$\begin{array}{c c} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

matrix A is called the vertex edge incidence matrix. or simply incidence matrix.

The incidence matrix contains only two elements 0 & 1. The incidence matrix is called a binary matrix. (without self loops) such a matrix is called a binary matrix.

The following observations about the incidence matrix A.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two 1's
2. The number of 1's in each row equals the degree of the corresponding vertex.
3. A row with all 0's therefore represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix.
5. If a graph G is disconnected & consists of two components g_1 & g_2. the incidence matrix $A(G)$ of graph G can be written in a block diagonal form as

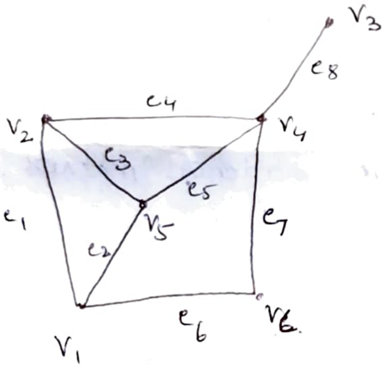
$$A(G) = \begin{bmatrix} A(g_1) & 0 \\ 0 & A(g_2) \end{bmatrix}$$

(6) Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices & edges of the same graph

Adjacency matrix

The adjacency matrix of a graph G with n vertices & no parallel edges is an n by n symmetric binary matrix. $X = [x_{ij}]$ defined over the ring of integers such that

$x_{ij} = 1$ if there is an edge between i^{th} & j^{th} vertices &
$= 0$ if there is no edge between them



$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A simple graph & its adjacency matrix

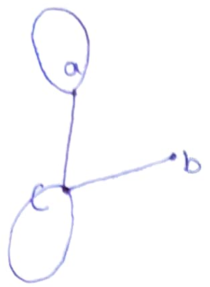
1. The entries along the principal diagonal of X are all 0's if and only if the graph has no self loops. A self loop at the i^{th} vertex corresponds to $x_{ii} = 1$
2. The definition of adjacency matrix makes no provision for parallel edges. The adjacency matrix X was defined for graphs without parallel edges.
3. If the graph has no self loops the degree of a vertex equals the number of 1's in the corresponding row or column of X.

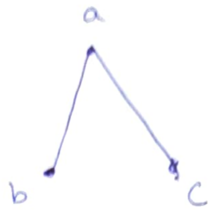
(b)

$$\begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

(ii)

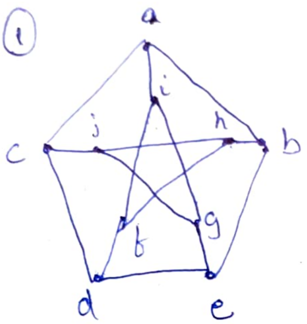
$$\begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$





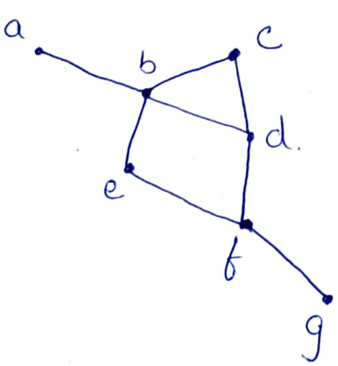
write the adjancency matrix for the following graphs

(1)



	a	b	c	d	e	f	g	h	i	j
a	0	1	1	0	0	0	0	0	1	0
b	1	0	0	0	1	0	0	1	0	0
c	1	0	0	1	0	0	0	0	0	1
d	0	0	1	0	1	1	0	0	0	0
e	0	1	0	1	0	0	1	0	0	0
f	0	0	0	1	0	0	0	1	1	0
g	0	0	0	0	1	0	0	0	1	1
h	0	1	0	0	0	1	0	0	0	1
i	1	0	0	0	0	1	1	0	0	0
j	0	0	1	0	0	0	1	1	0	0

(2)



	a	b	c	d	e	f	g
a	0	1	0	0	0	0	0
b	1	0	1	1	1	0	0
c	0	1	0	1	0	0	0
d	0	1	1	0	0	1	0
e	0	1	0	0	0	1	0
f	0	0	0	1	1	0	1
g	0	0	0	0	0	1	0