Vector space

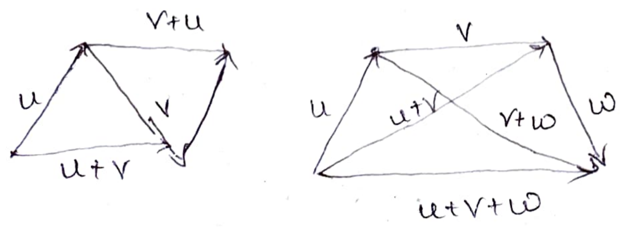
A vector space is a nonempty set V of objects called vectors on which are defined two operations called addition & multiplication by scalars (real numbers) subject to the ten axioms for all vectors u, v and w in V and for all scalars c and d.

1. The sum of u and v denoted by $u+v$ is in V
2. $u+v = v+u$.
3. $(u+v)+w = u+(v+w)$.
4. There is a zero vector 0 in V such that $u+0 = u$.
5. For each u in V there is a vector $-u$ in V such that $u+(-u) = 0$.
6. The scalar multiple of u by c denoted by cu is in V
7. $c(u+v) = cu+cv$
8. $(c+d)u = cu+du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

Note :- For each u in V & scalar c.

$$0u = 0$$
$$c0 = 0$$
$$-u = (-1)u.$$

Eg:- Let V be the set of all arrows in three dimensional space with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule and for each v in V, define cv to be the arrow whose length is $|c|$ times the length of v, pointing in the same direction as v if $c \geq 0$ and otherwise pointing in the opposite direction.



Definition

A subspace of a vector space V is a subset H of V that has tree properties

(a) The zero vector of V is in H^2.

(b) H is closed under vector addition. That is for each u & v in H, the sum u+v is in H

(c) H is closed under multiplication by scalars. That is for each u in H and each scalar c, the vector cu is in H.

Eg:- The set consisting of only the zero vector in a vector space V is a subspace of V called the zero subspace and written as {0}.

Linear Combination of vectors.

If one vector is equal to the sum of scalar multiples of other vectors, it is said to be a linear combination of the other vectors. $B = C_1V_1 + C_2V_2 + C_3V_3 + \cdots + C_pV_p$

A subspace spanned by a set.

A subspace Spann $\{v_1 \cdots v_p\}$ denotes the set of all vectors that can be written as linear combination of $v_1 \cdots v_p$.

Eg:- Given v_1 & v_2 in a vector space V, let $H = \text{span}\{v_1, v_2\}$ Show that H is a subspace of V

Solu The zero vector is in H since

$$0 = 0v_1 + 0v_2$$

H is closed under vector addition take two arbitrary vectors in H.

$$u = s_1v_1 + s_2v_2 \quad \& \quad w = t_1v_1 + t_2v_2$$

$$u + w = (s_1 v_1 + s_2 v_2) + (t_1 v_1 + t_2 v_2)$$
$$= (s_1 + t_1) v_1 + (s_2 + t_2) v_2$$

So $u + w$ is in H.

If c is any scalar

$$cu = c(s_1 v_1 + s_2 v_2) = (cs_1) v_1 + (cs_2) v_2$$

which show that cu is in H is closed under scalar multiplication. Thus H is a subspace of V.

Note :- If $v_1 \ldots v_p$ are in a vector space V, then span $\{v_1 \ldots v_p\}$ is a subspace of V.

Eg :- Express the vector $b = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of the vectors $v_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Solu :- we need to find numbers x_1, x_2, x_3 satisfying

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b.$$

The vector equation is equivalent to matrix equation.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} \qquad XV = b. \quad X[v_1, v_2\ v_3] = b.$$

Reduce the matrix to row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right] \qquad R_3 \leftrightarrow R_2 \qquad R_3 \to \tfrac{1}{2} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & -1 & 3 \end{array}\right] \qquad R_3 \to R_3 + 3R_1$$

10/2/ ICA

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

$$x_1 + x_2 + x_3 = 2$$

$$x_2 + x_3 = 4$$

$$5x_3 = 15 \Rightarrow x_3 = 3$$

$$x_2 + 3 = 4 \Rightarrow x_2 = 1$$

$$x_1 + 1 + 3 = 2 \Rightarrow x_1 = 2 - 4 = -2$$

$$x_1 = 1 \quad x_2 = -2 \;\&\; x_3 = 3$$

$$b = v_1 - 2v_2 + 3v_3$$

$$\begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-2+3 \\ 5\cancel{6}4+12 \\ -1-2+9 \end{bmatrix}$$

(2) write the vector $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

Solu

x_1 x_2 x_3 are real numbers

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 4 & -1 \end{array}\right] \quad R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & -2 & 0 & 3 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 8 & 1 \end{array}\right]$$

$$x_1 + 2x_2 + 2x_3 = 1$$

$$x_2 + 4x_3 = -1$$

$$-8x_3 = 1 \Rightarrow x_3 = 1/8$$

$$x_2 + 4\left(\tfrac{1}{8}\right) = -1$$

$$x_2 = -1 - \tfrac{1}{2} = -\tfrac{3}{2}$$

$$x_1 + 2x_2 + 2x_3 = 1$$

$$x_1 + 2\left(-\tfrac{3}{2}\right) + 2\left(\tfrac{1}{8}\right) = 1$$

$$x_1 = 4 - \tfrac{1}{4} = \tfrac{15}{4}$$

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \tfrac{15}{4} + 2\left(-\tfrac{3}{2}\right) + 2\left(\tfrac{1}{8}\right) \\ -2\left(-\tfrac{3}{2}\right) \\ -\tfrac{3}{2} + 4\left(\tfrac{1}{8}\right) \end{bmatrix}$$

$$b = \tfrac{15}{4} v_1 - \tfrac{3}{2} v_2 + \tfrac{1}{8} v_3$$

Null Space

Definition :- The null space of an $m \times n$ matrix A written as Null A is the set of all solutions to the homogeneous equation $Ax = 0$ In set notation.

$$\text{Null } A = \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}.$$

Eg :- Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and let $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ Determine if u belongs to the null space of A

Solu :- u satisfies $Au = 0$.

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus u is in Null A.

The Column Space of a matrix.

Definition The Column space of an $m \times n$ matrix A, written as Col$A$ is the set of all linear combinations of the Columns of A. If $A = [a_1, \dots a_n]$ then.

$$\text{Col}A = \text{Span}\{a_1 \dots a_n\}.$$

The Column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Eg:- $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a\ b \text{ in } \mathbb{R} \right\}$

$$W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$\text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$

Note:- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equations $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

Linear transformation: T from a vector space V into a vector space W is a rule that assigns to each vector x in V a unique vector $T(x)$ in W. such that. W = all the

(i) $T(u+v) = T(u) + T(v)$ $\forall$ u, v in V and.

(ii) $T(cu) = cT(u)$ $\forall$ $u \in V$ and all scalars c.

Theorem:- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \quad \forall\ x \text{ in } \mathbb{R}^n.$$

In fact A is the $m \times n$ matrix whose j^{th} column is the vector $T(e_j)$ where e_j is the j^{th} column of the identity matrix in $\mathbb{R}^n$

$$A = [T(e_1) \cdots T(e_n)]$$

$$Ax = [T(e_1) \cdots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Linearly independent sets & linearly dependent

A set of vectors $\{v_1 \ldots v_p\}$ in V is said to be linearly independent if the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

has only the trivial solution $c_1 = 0 \quad c_2 = 0 \ldots \quad c_p = 0$

The set $\{v_1 \ldots v_p\}$ is said to be linearly dependent if

$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$ has a nontrivial solution

That is if there are some weights $c_1, \ldots c_p$ not all zero. such that equ ① holds. This equation is called linear dependence relation among $v_1 \ldots v_p$.

Note :- A set containing a single vector v is linearly independent if and only if $v \neq 0$.

② A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.

③ A set containing the zero vector is linearly dependent.

Basis :-

Definition :- Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1 \ldots b_p\}$ in V is a basis for H if

(i) B is a linearly independent set, and

(ii) the subspace spanned by B coincides with H :

ie $H = \text{span}\{b_1 \ldots b_p\}$

~~eg :- let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$ $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$.~~

Basis applies

① $H = V$ ($\because$ vector space is a subspace of itself)

② A basis V is a linearly independent set that spans V.

③ $H \neq V$ includes the requirement that each of the vectors $b_1 \ldots b_p$ must belong to H.

The Dimension of a vector space

Def:- If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite dimensional.

Eg:- Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

Soln H is the set of all linear combinations of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

$v_1 \neq 0$ v_2 is not a multiple of v_1

v_3 is a multiple of v_2.

By the spanning set discard v_3

v_4 is not a linear Combination of v_1 & v_2

So $\{v_1, v_2, v_4\}$ is linearly independent

& hence is a basis for H Thus dim H = 3.

Standard Basis

The set of n vectors.

$$\{(1, 0 \ldots 0)\ (0, 1 \ldots 0) \ldots (0, \ldots 1)\}$$

is a basis for $\mathbb{R}^n$. This basis is called the standard basis for $\mathbb{R}^n$.

...ve that any set of vectors which contains the zero vector ... linearly dependent.

Solu: Let 0 be the zero vector and $v_1 . v_2 \dots V_K$ are the other vectors in the set.

Then we have the non trivial linear combination

$$1 \cdot 0 + 0 v_1 + 0 v_2 + \dots + 0 v_K = 0$$

This is a non trivial linear combination because one of the coefficients is non zero.

Thus by definition the set $\{0, v_1 v_2 \dots v_K\}$ is linearly dependent.

(4) For what values of a is the following set S linearly dependent?

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ a \end{bmatrix} \begin{bmatrix} a \\ 0 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a^2 \\ 7 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \\ a^3 \end{bmatrix} \right\}$$

Solu: Since the set S consist of five 4-dimensional vectors it is linearly dependent regardless of the value of a.

Thus for any value of a the set S is linearly dependent.

(5) Determine whether the following set of vectors linearly independent or linearly dependent. If the set is linearly dependent express one vector in the set as a linear combination of the other.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}$$

Solⁿ: Consider the linear combination.

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} = 0 \rightarrow (*)$$

with variables x_1 x_2 x_3 x_4

we determine whether there is $(x_1\ x_2\ x_3\ x_4) \neq (0,0,0,0)$ satisfying the above linear combination $(*)$

The linear combination $(*)$ is written as the matrix equation

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -2 \\ -1 & 3 & 0 & 7 \\ 0 & 4 & 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

To find the solutions of the equations we apply the $[A \mid 0]$ can be reduced by elementary row operation.

$$[A \mid 0] = \left[\begin{array}{cccc:c} 1 & 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ -1 & 3 & 0 & 7 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right] \quad \begin{array}{l} R_3 + R_1 \\ \frac{1}{2} R_2 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 4 & -1 & 5 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \rightarrow R_3 - 4R_2 \\ R_4 \rightarrow R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & 5 & 15 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \rightarrow \frac{1}{3} R_3 \\ R_4 \rightarrow \frac{1}{5} R_4 \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \quad R_4 \rightarrow R_4 - R_3$$

$$\left[\begin{array}{cccc:c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

solution is given by

$x_1 = x_4 \quad x_2 = -2x_4 \quad x_3 = -3x_4$ where x_4 is a free variable

If we take $x_4 = 1$. Then we have a non zero solution.

$x_1 = 1 \quad x_2 = -2 \quad x_3 = -3 \quad x_4 = 1$

Thus the set is linearly dependent.

Substituting the values into (*) we have

$$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - 3\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix} + \begin{bmatrix}-2\\-2\\7\\11\end{bmatrix} = 0$$

Solving for the last vector we obtain the linear combination.

$$\begin{bmatrix}-2\\-2\\7\\11\end{bmatrix} = -\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 3\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix} = \begin{bmatrix}-1+2-3\\0+4-6\\+1+6+0\\0+8+3\end{bmatrix}$$

(6) Find the value(s) of h for which the following set of vectors. $\left\{ V_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad V_2 = \begin{bmatrix}h\\1\\-h\end{bmatrix} \quad V_3 = \begin{bmatrix}1\\2h\\3h+1\end{bmatrix} \right\}$ is linearly independent.

Solu Let us consider the linear combination

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0.$$

If this homogeneous system has only zero solutions. $x_1 = x_2 = x_3 = 0$ Then the vectors $v_1\ v_2\ v_3$ are linearly independent.

we reduce the arugmented matrix for the system as follows.

$$\begin{bmatrix}1 & h & 1\\0 & 1 & 2h\\0 & -h & 3h+1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \xrightarrow{R_3 + hR_2} \left[\begin{array}{ccc:c}1 & h & 1 & 0\\0 & 1 & 2h & 0\\0 & 0 & 2h^2+3h+1 & 0\end{array}\right]$$

From this we see that the homogeneous system (1) has only the zero solution if and only if

$$2h^2+3h+1 \neq 0$$

Since we have $2h^2+3h+1 = (2h+1)(h+1)$

if $h \neq -1/2, -1$ then $2h^2+3h+1 \neq 0$

$\therefore$ the vectors v_1, v_2, v_3 are linearly independent for any h except $-1/2, -1$

(7) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ a \\ 5 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 4 \\ b \end{bmatrix}$ be vectors in R^3. Determine a condition on the scalars a, b so that the set of vectors $\{v_1, v_2, v_3\}$ is linearly dependent.

Solu Consider the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$

where 0 is the three dimensional zero vector.

Our goal is to find a condition on a, b so that the above equation has a non trivial solu x_1, x_2, x_3

This equation is equivalent to the 3×3 homogeneous system of linear equations.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & a & 4 \\ 0 & 5 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 2 & a & 4 & 0 \\ 0 & 5 & b & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & a-2 & 4 & 0 \\ 0 & 5 & b & 0 \end{array}\right] \quad \tfrac{1}{5}R_3$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & a-2 & 4 & 0 \\ 0 & 1 & b/5 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & a-2 & 4 & 0 \end{array}\right] \quad R_3 \to R_3 - (a-2)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & \frac{4-b(a-2)}{5} & 0 \end{array}\right]$$

Case (i)

If $\frac{4-b(a-2)}{5} = 0$ then we obtain matrix in echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This implies x_3 is a free variables hence homogeneous system has a non zero solution. x_1 x_2 x_3

Hence in this case the set $\{v_1\ v_2\ v_3\}$ is linearly dependent

Case (ii)

If $\frac{4-b(a-2)}{5} \neq 0$

than divide the third row by this number and obtain.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & b/5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

we obtain a solution $x_1 = x_2 = x_3 = 0$

Thus in this case the set $\{v_1, v_2, v_3\}$ is linearly independent.

Thus we conclude the set $\{v_1\ v_2\ v_3\}$ is linearly dependent iff $\frac{4-b(a-2)}{5} = 0$.

Thus the condition on a, b is $b(a-2) = 20$

Note:-

① $AX = B \Rightarrow AX = [0]$ is the matrix representation of the homogeneous set of equations where $[0]$ is null matrix.

$x_1 = x_2 \ldots = x_n = 0$ solution of the homogeneous system of equations is called a trivial solution.

If atleast one x_i $(i = 1, 2 \ldots n)$ is not equal to zero then is called non trivial solution.

[A : B] matrix

1. $\rho[A:B] = r \to$ rank, n being the numbers of unknowns.
   $= r = n \to$ unique solution.

2. $\rho[A:B] = r < n \to$ Infinite solution
   $(n-r)$ unknowns can take arbitrary values. $\rho[A] \neq \rho[A:B]$ implies that the system is inconsistent. 97

3. $\rho[A] \neq \rho[A:B]$ the system is inconsistent.

Span of set of vectors.

Definition :- If $S = \{v_1, v_2 \dots v_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combination of the vector in S.

$$\text{Span}(S) = \{c_1v_1 + c_2v_2 + \dots + c_kv_k \mid \forall c_i \in R$$

(the set of all linear combinations of vectors in S)

If every vector in a given vector space can be written as a linear combination of vectors in a given set S then S is called a spanning set of the vector space.

1(a) For what values of h will y be in Span $\{v_1, v_2, v_3\}$ if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \text{ \& } y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$y = x_1v_1 + x_2v_2 + x_3v_3$$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right] \quad R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-8+3 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right] \qquad h-5=0 \quad h=5$$

3 (a) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ & $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

(i) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

Ans No, 3 vectors.

(ii) How many vectors are in Span $\{v_1, v_2, v_3\}$?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} R_3 \to R_3 + R_1 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & 7 \end{bmatrix} \qquad \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right] R_3 \to \tfrac{1}{5}R_3 \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad r=2 \quad n=3 \quad r<n \text{ Consistent.}$$

and has infinitely many solution.

(iii) x_3 is a free variable. $x_3 = 1$ $\quad x_2 + 2x_3 = 1$ $\quad x_2 = 1-2 = -1$

$x_1 + 2x_2 + 4x_3 = 3 \Rightarrow x_1 - 2 + 4 = 3$ $\quad x_1 = 3-2 = 1$

Note :- The subspace of all linear combination of the set of given vector space is called the subspace generated by these vectors or spanned by these vectors

(2) The subspace spanned by any non-zero vector α of a vector space V consists of all scalar multiples of α. Geometrically it represents a line through the origin and α

(3) The subspace spanned by any two non zero vectors α & β which are not multiples of each other represents a plane passing through the origin α & β.

$w = 1v_1 - v_2 + v_3$ $\quad$ w is a subspace spanned by $\{v_1, v_2, v_3\}$

$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1-2+4 \\ 0-1+2 \\ -1-3+6 \end{bmatrix}$$

4 (b) Find a spanning set for the null space of matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

Soln

$$Ax = 0$$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$[A:0]$

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \quad R_2 \to \tfrac{1}{5} R_2$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r = 2 \\ n = 5 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$
$$x_3 + 2x_4 - 2x_5 = 0$$
$$0 = 0$$

x_2, x_4, x_5 are free variables.

$x_1 = 2x_2 + x_4 - 3x_5 \qquad x_3 = -2x_4 + 2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

linear combination of u v and w is an element of Nul A

Thus $\{u, v, w\}$ is a spanning set for Nul A.

5 (a) Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ Let $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(a) Determine if u is in Nul A. Could u be in Col A?

(b) Determine if v is in Col A could v be in Nul A?

Solu

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u is not a solution of $Ax = 0$, so u is not a Nul A.

Also with four entries u could not possibly be in Col A.

since Col A is a subspace of $\mathbb{R}^3$

(b) $[A \ v]$

$$= \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 2 & 10 & 9 & -3 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & +1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right]$$

It is clear that the equation $Ax = v$ is consistent so

v is in Col A with only three entries, v could not possibly

be in Nul A since Nul A is subspace of $\mathbb{R}^4$

3(b) Show that w is in the subspace R^4 spanned by $v_1\ v_2\ v_3$ where.

(b)
$$\begin{array}{cccc} v_1 & v_2 & v_3 & w \end{array}$$
$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -1 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{array}\right]$$

$R_2 \to R_2 + \frac{4}{7} R_1$

$R_3 \to R_3 + \frac{2}{7} R_1$

$R_4 \to R_4 - \frac{9}{7} R_1$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & \frac{19}{7} & \frac{8}{7} & \frac{13}{7} \\ 0 & -\frac{15}{7} & \frac{10}{7} & \frac{10}{7} \\ 0 & -\frac{13}{7} & \frac{32}{7} & \frac{137}{7} \end{array}\right]$$

$R_2 \to 7R_2$

$R_3 \to 7R_3$

$R_4 \to 7R_4$

$R_3 \to \frac{1}{5} R_3$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 19 & -8 & 13 \\ 0 & -13 & 32 & 137 \end{array}\right]$$

$R_3 \to R_3 + \frac{19}{3} R_2$

$R_4 \to R_4 - \frac{13}{3} R_2$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & \frac{14}{3} & \frac{77}{3} \\ 0 & 0 & \frac{70}{3} & \frac{385}{3} \end{array}\right]$$

$R_3 \to 3R_3$

$R_4 \to 3R_4$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 70 & 385 \end{array}\right]$$

$R_4 \to R_4 - \frac{70}{14} R_3$

$$\left[\begin{array}{ccc:c} 7 & -4 & -9 & -9 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 14 & 77 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

consistence.

$7x_1 - 4x_2 - 9x_3 = -9.$

$-3x_2 + 2x_3 = 2$

$14x_3 = 77$

$x_3 = 5.5$

$-3x_2 + 2(5.5) = 2$

$-3x_2 = 2 - (5.5)2$

$-3x_2 = -9$

$x_2 = 3$

$7x_1 - 4(3) - 9(5.5) = -9.$

$7x_1 = -9 + 12 + 49.5$

$x_1 = \frac{52.5}{7} = 7.5$

Find a basis for span (S) where $S = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}-1\\-2\\-1\end{bmatrix} \begin{bmatrix}2\\6\\-2\end{bmatrix} \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$

Solu

$$x_1\begin{bmatrix}1\\2\\1\end{bmatrix} + x_2\begin{bmatrix}-1\\-2\\-1\end{bmatrix} + x_3\begin{bmatrix}2\\6\\-2\end{bmatrix} + x_4\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 2 & 2 & 6 & 1 & 0 \\ 1 & -1 & -2 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -4 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_1 \to R_1 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to \tfrac{1}{2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the above matrix has leading 1's in first and third columns we can conclude the first and third vector of S form a basis of span (S).

$\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}2\\6\\-2\end{bmatrix} \right\}$ is a basis for span (S).

(12) Let $S = \{V_1, V_2, V_3, V_4, V_5\}$ where

$$V_1 = \begin{bmatrix}1\\2\\2\\-1\end{bmatrix} \quad V_2 = \begin{bmatrix}1\\3\\1\\1\end{bmatrix} \quad V_3 = \begin{bmatrix}1\\5\\-1\\5\end{bmatrix} \quad V_4 = \begin{bmatrix}1\\1\\4\\-1\end{bmatrix} \quad V_5 = \begin{bmatrix}2\\7\\0\\2\end{bmatrix}$$

Find a basis for the span (S)

Solu

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 1 & 7 \\ 2 & 1 & -1 & 4 & 0 \\ -1 & 1 & 5 & -1 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & -1 & -3 & 2 & -4 \\ 0 & 2 & 6 & 0 & 4 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + R_3 \\ R_4 \to R_4 - 2R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that 1st 2nd & 4th column vectors of ... contain the leading 1 entries Hence the 1st 2nd & 4th column vectors of A form a basis of span(s)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$ is a basis for span(s)

9(b) Find the dimension of the subspace spanned by the given vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}$ $v_4 = \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

Solu :-

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 5 & -20 & -15 \end{bmatrix} \quad R_3 \to \tfrac{1}{5} R_3$$

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 1 & -4 & -3 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

1st & 2nd column are basis

span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ Dim $V = 2$

that if a vector space V has a basis $B = \{b_1, b_2 \ldots b_n\}$ then any set in V containing more than n vectors must be linearly dependent.

Solu Let $\{u_1 \ldots u_p\}$ be a set in V with more than n vectors. The coordinate vectors $[u_1]_B \ldots [u_p]_B$ form a linearly dependent set in $\mathbb{R}^n$ because there are more vectors (p) than entries (n) in each vector, so $\exists$ scalars $c_1 \ldots c_p$ not all zero such that

$$c_1[u_1]_B + \cdots + c_p[u_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the coordinate mapping is a linear combination

$$[c_1u_1 + \cdots + c_pu_p]_B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector on the right contains the n weights needed to build the vector $c_1u_1 + \cdots + c_pu_p$ from the basis vectors in B. ie $c_1u_1 + \cdots + c_pu_p = 0 \cdot b_1 + \cdots + 0 \cdot b_n = 0$

Since the c_i are not all zero $\{u_1, u_2 \ldots u_p\}$ is linearly dependent

This implies that if a vector space V has a basis $B = \{b_1 \ldots b_n\}$ then each linearly independent set in V has no more than n vectors.

10(a) Let $T: R^2 \to R^3$ be a linear transformation. Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ be 2 dimensional vectors. Suppose that $T(u) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and $T(v) = T\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Let $w = \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$. Find the formula for $T(w)$ in terms of x & y.

Solu

$$w = au + bv$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + 3b \\ 2a + 5b \end{bmatrix}$$

$x = a + 3b \quad \times 2 \Rightarrow \quad 2x = 2a + 6b$

$y = 2a + 5b \qquad\qquad - y = -2a \mp 5b$

$\qquad\qquad\qquad\qquad\qquad 2x - y = b$

$x = a + 3(2x - y)$

$x = a + 6x - 3y \Rightarrow a = 3y - 5x$

$w = au + bv \Rightarrow T(w) = T(au + bv)$

$T(w) = aT(u) + bT(v)$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = aT\begin{bmatrix} 1 \\ 2 \end{bmatrix} + bT\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} -3 \\ 5 \end{bmatrix} + b\begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3a + 7b \\ 5a + b \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3(3y - 5x) + 7(2x - y) \\ 5(3y - 5x) + (2x - y) \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9y + 15x + 14x - 7y \\ 15y - 25x + 2x - y \end{bmatrix} = \begin{bmatrix} 29x - 16y \\ -23x + 14y \end{bmatrix}$$

(b) Let $T: R^2 \to R^3$ be linear transformation such that $T\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $T\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$ Find the matrix representation of T (with respect to the standard basis for $\mathbb{R}^3$)

$e_1 = au + bv$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 3 \\ 2 \end{bmatrix} + b\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 2 & 3 & 0 \end{array}\right] \quad R_2 \to R_2 - \frac{2}{3}R_1$$

$3 - \frac{2}{3}(4) = \frac{9-8}{3}$

$0 - \frac{2}{3}(1)$

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 0 & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

$3a + 4b = 1 \qquad\qquad 3a + 4(-2) = 1$

$b\frac{1}{3} = -\frac{2}{3} \qquad\qquad 3a = 1 + 8$

$\qquad\qquad\qquad\qquad 3a = 9$

$b = -2 \qquad\qquad\qquad a = 3$

$e_2 = au + bv$

$$\begin{bmatrix}0\\1\end{bmatrix} = a\begin{bmatrix}3\\2\end{bmatrix} + b\begin{bmatrix}4\\3\end{bmatrix} \qquad T(e_1) = aT(u) + bT(v)$$

$$\begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\left[\begin{array}{cc:c}3 & 4 & 0\\2 & 3 & 1\end{array}\right] \qquad R_2 \to R_2 - \tfrac{2}{3}R_1$$

$$3 - \tfrac{2}{3}(4) = \frac{9-8}{3}$$

$$1 - \tfrac{2}{3}(0)$$

$$\left[\begin{array}{cc:c}3 & 4 & 0\\0 & \tfrac{1}{3} & 1\end{array}\right]$$

$3a + 4b = 0$

$\tfrac{1}{3}b = 1 \qquad b = 3$

$3a + 4(3) = 0$

$3a = -12$

$a = -4$

$$A = \{T(e_1),\ T(e_2)\}$$

$$A = \left\{3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\-5\\1\end{bmatrix},\ -4\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}0\\-5\\1\end{bmatrix}\right\}$$

$$A = \begin{bmatrix}3-0 & -4+0\\6+10 & -8-15\\9-2 & -12+3\end{bmatrix} = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}$$

(12) Alternate form of finding the basis.

$$A = \begin{bmatrix}1 & 1 & 1 & 1 & 2\\2 & 3 & 5 & 1 & 7\\2 & 1 & -1 & 4 & 0\\-1 & 1 & 5 & -1 & 2\end{bmatrix}$$

$$A^T = \begin{bmatrix}1 & 2 & 2 & -1\\1 & 3 & 1 & 1\\1 & 5 & -1 & 5\\1 & 1 & 4 & -1\\2 & 7 & 0 & 2\end{bmatrix} \qquad \begin{array}{l}R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\\R_4 \to R_4 - R_1\\R_5 \to R_5 - 2R_1\end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 3 & -4 & 4 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + R_2 \\ R_5 \to R_5 - 3R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & +2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad R_5 \to R_5 + R_4$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The set of non zero rows is & their corresponding vector

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 1 \end{bmatrix} \right\}$$ is a basis for span (S)

5 (b) let $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix}$ For each of the following vectors determine whether the vectors are in the null space $N(A)$

(a) $\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then describe the null space $N(A)$ of the matrix A.

Soln: Null space is $AX = 0$

(a) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+0+3+0 \\ 0+0+1+0 \\ -3+0+4+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not a Null space

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4+0+6-2 \\ 0-3+2+1 \\ -4-3+8-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ is a $N(A)$

(c) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ is a $N(A)$

(d) $$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 3 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- the size of matrix A is 3×4 & x is 3×1 matrix multiplication is not possible $\therefore$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ vector is not in $N(A)$.