

& y-axis is called the axis of imaginaries.

2011
July

The plane on which the complex numbers are represented is called

DAY 190-175 • Week • 27

SATURDAY

09

the complex plane or Argand plane.
diagram.

8:00

C5. Complex Differentiation

9:00

Preliminaries:

10:00

1. Let x & y be any two real numbers then $Z = x + iy$ is the complex number. Here x is the real part & y is the imaginary part of Z .

11:00

2. If $Z = x + iy$ is a complex number, then the conjugate of Z is given by $\bar{Z} = x - iy$.

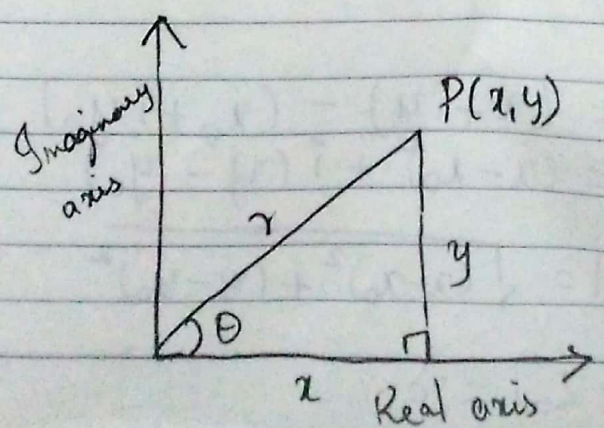
2:00

3. Geometrical representation of complex number
For complex no: $Z = x + iy$ is represented by the point $P(x, y)$ in the co-ordinates axis.

4:00

The x -axis is referred to as real axis & y -axis is referred as imaginary axis & the plane on which the complex numbers are represented is called complex plane / Z -plane / Argand plane.

Evening



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Thu	6	13	20	27		Thu	10	17	24	31
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	3	10	17	24	31		14	21	28	

From Diagram, $x = r \cos \theta$, $y = r \sin \theta$

10 $|z| = r = \sqrt{x^2 + y^2}$
SUNDAY $\theta = \tan^{-1}(y/x)$

10-07-2011

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DAY 191-174 • Week 27

Appointment where, r is called the modulus of z
8:00 θ is called the amplitude or argument
of z .

9:00 We have $z = x + iy$
 $z = r (\cos \theta + i \sin \theta)$

10:00

$z = r e^{i\theta}$ polar form of z

11:00

12:00 $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

1:00

$\cos(ix) = \frac{e^{-x} + e^x}{2} = \cosh x$

2:00

$\sin(ix) = \frac{e^{-x} - e^x}{2i} = i \sinh x$

3:00

4:00 5. Neighbourhood of pt. z_0 : It is a set
5:00 of all points z such that $|z - z_0| \leq \delta$
where δ is a very small real
positive number.

6. Geometrical Meaning:

Evening

Let $z = x + iy$ be any complex number
& $z_0 = x_0 + iy_0$

Consider $z - z_0 = (x + iy) - (x_0 + iy_0)$
 $= (x - x_0) + i(y - y_0)$

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Wed	4	11	18	25		Wed	1	8	15	22	29
Thu	5	12	19	26		Thu	2	9	16	23	30
Fri	6	13	20	27		Fri	3	10	17	24	
Sat	7	14	21	28		Sat	4	11	18	25	
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$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

Let $|z - z_0| = \delta$

Limit, Continuity & Differentiability for a 12 complex value.

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TUESDAY

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DAY 193, 172 • Week 26

Limit of a complex valued funⁿ

Appointment

Let $f(z) = w$ be complex value funⁿ,
defined in the neighbourhood of pt. z_0
is said to have limit l as $z \rightarrow z_0$,
if $\forall \epsilon > 0$ \mathbb{R} , there exist positive
real number, $\delta > 0$ such that
 $|f(z) - l| < \epsilon$ whenever $|z - z_0| < \delta$

ie. $\lim_{z \rightarrow z_0} f(z) = l$.

12:00

Continuity of a function:-

1:00

A complex valued function is said to
be continuous at $z = z_0$ if
i) $f(z)$ is defined at z_0
ii) $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

Differentiability:

A complex valued function $w = f(z)$
is said to be differentiable at $z = z_0$ if

$\lim_{\delta z \rightarrow 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$ exists & is unique.

Evening

MAY					JUNE						
Wk	17	18	19	20	21	Wk	22	23	24	25	26
Mon	30	2	9	16	23	Mon	6	13	20	27	
Tue	31	3	10	17	24	Tue	7	14	21	28	
Wed		4	11	18	25	Wed	1	8	15	22	29
Thu		5	12	19	26	Thu	2	9	16	23	30
Fri		6	13	20	27	Fri	3	10	17	24	
Sat		7	14	21	28	Sat	4	11	18	25	
Sun		1	8	15	22	29	Sun	5	12	19	26

Analytic Function / Regular $f(z)$ / Holomorphic

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WEDNESDAY

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A $f(z)$ $w = f(z)$ is said to be analytic $f(z)$ at pt $z = z_0$ if

the derivative of the $f(z)$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} \text{ exists } \&$$

unique at z_0 and also in the neighbourhood of point z_0 .

* Theorem: Cauchy - Riemann equation in Cartesian form

Statement: The necessary condition that a complex valued function $w = f(z) = u(x, y) + i v(x, y)$ may be analytic at any pt: $z = x + iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ & the the eq^{ns} known,

as Cauchy - Riemann eq^{ns} must be satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}$$

Proof: Let $f(z)$ be analytic at a point $z = x + iy$ & hence by the definition

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} \text{ exists } \& \text{ is unique}$$

In the cartesian form $f(z) = u(x, y) + i v(x, y)$ & let δz be the increment in z corresponding to the increments $\delta x, \delta y$ in x, y .

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{[u(x + \delta x, y + \delta y) + i v(x + \delta x, y + \delta y)] - [u(x, y) + i v(x, y)]}{\delta z}$$

JULY							AUGUST						
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Wed	11	12	13	14	15	16	Wed	13	14	15	16	17	18
Thu	18	19	20	21	22	23	Thu	20	21	22	23	24	25
Fri	25	26	27	28	29	30	Fri	27	28	29	30	31	
Sat	2	3	4	5	6	7	Sat	3	4	5	6	7	8
Sun	9	10	11	12	13	14	Sun	10	11	12	13	14	15

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(x+\delta x, y+\delta y) - u(x, y)}{\delta z} +$$

$$i \frac{v(x+\delta x, y+\delta y) - v(x, y)}{\delta z} \quad \text{--- (1)}$$

Now $\delta z = \delta x + i\delta y$, $z = x + iy$

$$= [(x+\delta x) + i(y+\delta y)] - (x+iy)$$

$$\rightarrow \delta z = \delta x + i\delta y$$

Since δz tends to zero, we have the following two possibilities

Case 1 Let $\delta y = 0$ so that $\delta z = \delta x$, $\delta z \rightarrow 0 \Rightarrow \delta x \rightarrow 0$

$$\textcircled{1} \Rightarrow f'(z) = \lim_{\delta x \rightarrow 0} \frac{u(x+\delta x, y) - u(x, y)}{\delta x} +$$

$$i \frac{v(x+\delta x, y) - v(x, y)}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Case 2 Let $\delta x = 0$, $\delta z = i\delta y$, $\delta z \rightarrow 0 \Rightarrow \delta y \rightarrow 0$

$$\textcircled{1} \Rightarrow f'(z) = \lim_{\delta y \rightarrow 0} \frac{u(x, y+\delta y) - u(x, y)}{i\delta y} +$$

$$i \frac{v(x, y+\delta y) - v(x, y)}{i\delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

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Mon	30	2	9	16	23	Mon	6	13	20	27	
Tue	31	3	10	17	24	Tue	7	14	21	28	
Wed	4	11	18	25		Wed	1	8	15	22	29
Thu	5	12	19	26		Thu	2	9	16	23	30
Fri	6	13	20	27		Fri	3	10	17	24	
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Equating R.H.S of (2) & (3),

FRIDAY

Appointment

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Equating the real & imaginary parts, we get

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \rightarrow \text{C-R eq}^{\text{ns}}$$

Theorem 2: [Cauchy-Riemann equations in the polar form]

If $f(z) = f(re^{i\theta}) = u(r, \theta) + i v(r, \theta)$ is analytic at a point z , then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ & satisfy the eq^{ns}:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Proof: $f(z) = u(r, \theta) + i v(r, \theta)$

Let $f(z)$ be analytic at a point $z = re^{i\theta}$

By the defⁿ, $f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$ exists & is unique.

$$f(z) = u(r, \theta) + i v(r, \theta)$$

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{u(r + \delta r, \theta + \delta \theta) + i v(r + \delta r, \theta + \delta \theta)}{\delta z}$$

$$- [u(r, \theta) + i v(r, \theta)]$$

$$= \lim_{\delta z \rightarrow 0} \frac{u(r + \delta r, \theta + \delta \theta) - u(r, \theta)}{\delta z}$$

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Wed	4	11	18	25	31	Wed	3
Thu	6	13	20	27		Thu	4
Fri	1	14	21	28		Fri	5
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We have $x = r \cos \theta$

$y = r \sin \theta$

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SATURDAY

16.07.2011

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$$w = f(z) \rightarrow (u, v) \rightarrow (x, y) \rightarrow (r, \theta)$$

Appointment

$$(u, v) \rightarrow (x, y) \rightarrow (r, \theta)$$

8.00

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

9.00

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \quad \text{--- (1)}$$

10.00

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta) \quad \text{--- (2)} \end{aligned}$$

11.00

12.00

1.00

$$\frac{\partial u}{\partial \theta} = r \left[-\sin \theta \frac{\partial u}{\partial x} + \cos \theta \frac{\partial u}{\partial y} \right] \quad \text{--- (2)}$$

2.00

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r}$$

3.00

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta \quad \text{--- (3)}$$

4.00

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta) \quad \text{--- (4)}$$

5.00

Now consider (3),

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta$$

Evening

Since $f(z)$ is analytic, C-R eq^{ns} in Cartesian form

$$u_x = v_y, \quad v_x = -u_y$$

MAY

JUNE

Wk	17	18	19	20	21
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Wed	4	11	18	25	
Thu	5	12	19	26	
Fri	6	13	20	27	
Sat	7	14	21	28	
Sun	1	8	15	22	29

Wk	22	23	24	25	26
Mon	6	13	20	27	
Tue	7	14	21	28	
Wed	1	8	15	22	29
Thu	2	9	16	23	30
Fri	3	10	17	24	
Sat	4	11	18	25	
Sun	5	12	19	26	

$$\therefore \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta$$

Using eq^{ns} (2)

$$\frac{\partial v}{\partial r} = -r \frac{\partial u}{\partial x}$$

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$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

SUNDAY

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Appointment

Consider, (4)

$$800 \quad \frac{\partial v}{\partial \theta} = -r \left[\sin \theta \frac{\partial v}{\partial x} - \cos \theta \frac{\partial v}{\partial y} \right]$$

$$900 \quad = -r \left[-\sin \theta \frac{\partial u}{\partial y} - \cos \theta \frac{\partial u}{\partial x} \right]$$

$$1100 \quad = r \left[\frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right]$$

$$1200 \quad \frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r} \quad \left\{ \text{from (1)} \right\}$$

$$1400 \quad \Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}}$$

Note:

400 (1) Let $w = f(z)$ be analytic funⁿ then
by C-R eqⁿ in Cartesian form

$$500 \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

by C-R eqⁿ in polar form

$$\text{Evening} \quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

$\nabla^2 u(x_0)$ is proportional to the average of $u(z) - u(z_0)$ in a small ball around z_0 . $\nabla^2 u = 0$
 $\nabla^2 u = 0$ means that at every pt, u would be equal to the average of its neighbouring points.

		AUGUST										
Mon	26	27	28	29	30	Mon	31	1	2	3	4	5
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Wed	5	12	19	26		Wed	2	9	16	23	30	
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Sun	2	9	16	23	30	Sun	6	13	20	27		
						Sun	7	14	21	28		