& y-axis is called the axis of imaginaries.
The plane on which the complex no. are represented is called the complex plane or Argand plane diagram.

C5. Complex Differentiation

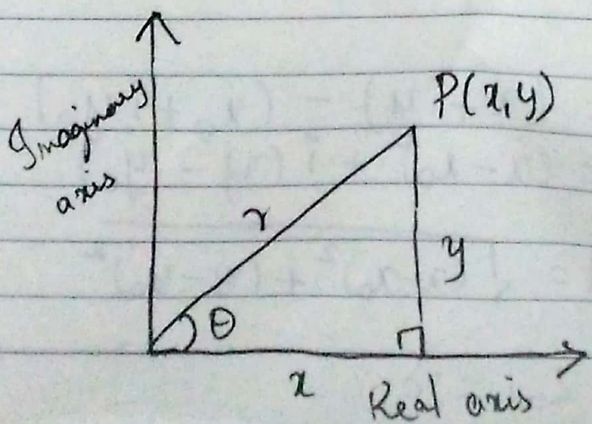
Preliminaries:

1. Let x & y be any two real numbers then $z = x+iy$ is the complex number. Here x is the real part & y is the imaginary part of z.

2. If $z = x+iy$ is a complex number, then the conjugate of z is given by
$\bar{z} = x - iy$.

3. Geometrical representation of complex number
For complex no.: $z = x+iy$ is represented by the point $P(x,y)$ in the co-ordinates axis.

The x-axis is referred as real axis & y-axis is referred as imaginary axis & the plane on which the complex numbers are represented is called complex plane / z-plane / Argand plane



From Diagram, $x = r\cos\theta$, $y = r\sin\theta$

$|z| = r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

where, r is called the modulus of z
θ is called the amplitude or argument of z.

We have $z = x+iy$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$ polar form of z.

~~At~~. $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$\cos(ix) = \dfrac{e^{-x}+e^{x}}{2} = \cosh x$

$\sin(ix) = \dfrac{e^{-x}-e^{x}}{2i} = i\sinh x$.

5. <u>Neighbourhood</u> <u>of pt. z_0</u>: It is a set of all points z such that $|z-z_0| \le \delta$ where δ is a very small real positive number.

6. <u>Geometrical Meaning</u>:
Let $z = x+iy$ be any complex number & $z_0 = x_0+iy_0$

Consider $z - z_0 = (x+iy) - (x_0+iy_0)$

$= (x-x_0) + i(y-y_0)$

$|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$

Let $|z-z_0| = \delta$

MAY					
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JUNE					
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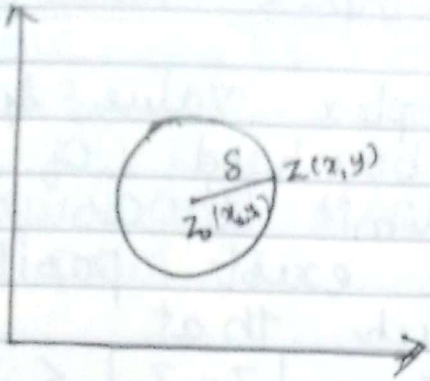
$$\delta = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\boxed{\delta^2 = (x-x_0)^2 + (y-y_0)^2}$$



$|z - z_0| = \delta$, represents the circle with radius δ and centre at $z_0(x_0, y_0)$.

Function of a Complex Variable:

A complex valued function $w = f(z)$ can be expressed in cartesian and polar form as follows

a) $w = f(z) = u(x,y) + i\, v(x,y)$.

Ex: $w = z^2 = (x+iy)^2$
$= x^2 - y^2 + 2ixy$
$= \underbrace{(x^2-y^2)}_{u} + i\,\underbrace{(2xy)}_{v}$.

u is the real part, v is the imaginary part.

b) Polar form: $w = f(z) = u(r,\theta) + i\,v(r,\theta)$

Ex: $w = z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta}$

$= r^2 [\cos 2\theta + i \sin 2\theta]$

$= \underbrace{r^2 \cos 2\theta}_{u} + i\, \underbrace{r^2 \sin 2\theta}_{v}$.

Limit, Continuity & Differentiability for a complex value.

Limit of a complex valued fu^n

Let $f(z) = w$ be complex value fu^n, defined in the neighbourhood of pt. z_0 is said to have limit 'l' as $z \to z_0$, if $\forall \epsilon > 0$ (small), there exist positive real number, $\delta > 0$ such that
$|f(z) - l| < \epsilon$ whenever $|z - z_0| < \delta$

ie. $\lim_{z \to z_0} f(z) = l$.

Continuity of a function:-

A complex valued function is said to be continuous at $z = z_0$ if
i) $f(z)$ is defined at z_0
ii) $\lim_{z \to z_0} f(z) = f(z_0)$.

Differentiability:

A complex valued function $w = f(z)$ is said to be differentiable at $z = z_0$ if

$\lim_{\delta z \to 0} \dfrac{f(z_0 + \delta z) - f(z_0)}{\delta z}$ exists & is unique.

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Analytic Function / Regular Fu^n / Holomorphic fu^n

A fu^n $w = f(z)$ is said to be analytic fu^n at pt $z = z_0$ if the derivative of the fu^n

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists & unique at z_0 and also in the neighbourhood of point z_0.

* **Theorem: Cauchy - Riemann equation in Cartesian form**

Statement: The necessary condition that a complex valued function $w = f(z) = u(x,y) + i\,v(x,y)$ may be analytic at any pt $z = x + iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ & the the eq^{ns} known as Cauchy - Riemann eq^{ns} must be satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof: Let $f(z)$ be analytic at a point $z = x+iy$ & hence by the definition

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists & is unique

In the cartesian form $f(z) = u(x,y) + i\,v(x,y)$ & let δz be the increment in z corresponding to the increments $\delta x, \delta y$ in x, y

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)] - [u(x,y) + i\,v(x,y)]}{\delta z}$$

Appointment

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + \frac{i\,[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \quad \text{--- (1)}$$

Now $\delta z = \cancel{\delta x}\ z + \delta z - z$, $z = x+iy$

$$= [(x+\delta x) + i(y+\delta y)] - (x+iy)$$

$$\rightarrow \delta z = \delta x + i\,\delta y.$$

Since δz tends to zero, we have the following two possibilities

Case 1) Let $\delta y = 0$ so that $\delta z = \delta x$, $\delta z \to 0 \Rightarrow \delta x \to 0$.

$$(1) \Rightarrow f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\,\frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Case 2) Let $\delta x = 0$, $\delta z = i\,\delta y$, $\delta z \to 0 \Rightarrow \delta y \to 0$

$$(1) \Rightarrow f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\,\delta y} + i\,\frac{v(x, y+\delta y) - v(x,y)}{i\,\delta y}$$

$$= -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Evening

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$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{---(3)}$$

Equating R.H.S of (2) & (3),

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating the real & imaginary parts, we get

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \rightarrow \text{C-R eq}^{ns}.$$

Theorem 2: [Cauchy-Riemann equations in the polar form]

If $f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ is analytic at a point z, then there exists four continous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ & satisfy the eq^{ns}:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof:

Let $f(z)$ be analytic at a point $z = re^{i\theta}$ $= u(r,\theta) + i\,v(r,\theta)$.

By the def^n, $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists & is unique.

$$f(z) = u(r,\theta) + i\,v(r,\theta)$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) + i\,v(r+\delta r, \theta+\delta\theta)}{\delta z} - \frac{[u(r,\theta) + i\,v(r,\theta)]}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i\frac{[v(r+\delta r, \theta+\delta\theta) - v(r,\theta)]}{\delta z}$$

We have $x = r\cos\theta$

$y = r\sin\theta$

$$w = f(z) \longrightarrow (u,v) \longrightarrow (x,y) \longrightarrow (r,\theta)$$

Appointment

$$(u,v) \longrightarrow (x,y) \longrightarrow (r,\theta)$$

8:00

9:00
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

10:00
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{--- (1)}$$

11:00
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

12:00
$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \quad \cancel{-e2}$$

1:00
$$\frac{\partial u}{\partial \theta} = r\left[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right] \quad \text{--- (2)}$$

2:00
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

3:00
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \quad \text{--- (3)}$$

4:00

5:00
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \quad \text{--- (4)}$$

Now consider (3),

Evening
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

Since $f(z)$ is analytic, C-R eq^{ns} in cartesian form

$$u_x = v_y \quad , \quad v_x = -u_y$$

$$\therefore \frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

Using eq^{ns} (2)

$$\frac{\partial v}{\partial r} = \cancel{-r\frac{\partial u}{\partial x}}$$

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$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

Consider, (4)

$$\frac{\partial v}{\partial \theta} = -r \left[\sin\theta \frac{\partial v}{\partial x} - \cos\theta \frac{\partial v}{\partial y} \right]$$

$$= -r \left[-\sin\theta \frac{\partial u}{\partial y} - \cos\theta \frac{\partial u}{\partial x} \right]$$

$$= r \left[\frac{\partial u}{\partial x} \cos\theta + \frac{\partial u}{\partial y} \sin\theta \right]$$

$$\frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r} \quad \{\text{from } (1)\}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}}$$

Note:

(1) Let $w = f(z)$ be analytic fu^n then

by C-R eq^n in cartesian form

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

by C-R eq^n in polar form

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

$\nabla^2 u(x_0)$ is proportional to the average of $u(x) - u(x_0)$ in a small ball around x_0. $\nabla^2 u = 0$

$\nabla^2 u = 0$ means that ~~at~~ every pt/ u would ~~be~~ equal to the average of its neighbouring ~~bo~~ points.