Complex Variables - I

Complex number :-

A number of the form $z = x + iy$ where x, y are real numbers and $i = \sqrt{-1}$ (or) $i^2 = -1$ is called a complex number.

x is called the real part of z and

y is called the imaginary part of z.

* $\bar{z} = x - iy$ is called the complex conjugate of z.
* $e^{ix} = \cos x + i\sin x$ & $e^{-ix} = \cos x - i\sin x$.
* $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$
* $\cosh x = \dfrac{e^{-x} + e^{x}}{2}$ and $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$
* $\cos(ix) = \cosh x$ and $\sin(ix) = i\sinh x$.
* De-Moivre's theorem:-

  $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, n is a real no
* polar form of z, $z = re^{i\theta}$

$|z| = r = \sqrt{x^2 + y^2}$ is called the Modulus of z.

and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ is called the amplitude of z (or) augument of z. denoted by $\text{amp}\, z$ (or) $\arg z$.

* properties :-

(a) $|z_1 . z_2| = |z_1| . |z_2|$

(b) $\text{amp}(z_1 . z_2) = \text{amp}\, z_1 + \text{amp}\, z_2$

(c) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(d) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\, z_1 - \text{amp}\, z_2$

(e) $|z_1 + z_2| \le |z_1| + |z_2|$

(f) $|z_1 - z_2| \ge |z_1| - |z_2|$

Neighbourhood:-

A nhd of a point z_0 in the complex plane is the set of all points z such that $|z-z_0| < \delta$ where δ is a small positive real number.

* $(x-x_0)^2 + (y-y_0)^2 = \delta^2$

This represents a circle with centre (x_0, y_0) and radius δ.

Geometrically a nhd of a point is the set of all points inside a circle having z_0 as the centre & δ as the radius.

Functions of a complex Variable :-

If it is possible to find one (or) more complex numbers ω for every value z. in a certain domain D we say that ω is a function of z defined for the domain D. In otherwords $\omega = f(z)$ is called a function of the complex variable z.

ω is said to be single-valued (or) many valued function of z according as for a given value of z. there corresponds one (or) more than one value of ω.

$\because \; z = x+iy \quad (\text{or}) \quad z = re^{i\theta}$

$\omega = f(z) = u(x,y) + iv(x,y)$ [Cartesian form]

$\omega = f(z) = u(r,\theta) + iv(r,\theta)$ [polar form]

Example:-

1) $f(z) = z^2$

i.e, $u+iv = (x+iy)^2 = x^2 + 2xiy + i^2y^2$

$= (x^2-y^2) + i(2xy)$

$\therefore u = x^2 - y^2$ & $v = 2xy$ in the. cartesian form.

$$f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{2i\theta}$$
$$= r^2[\cos 2\theta + i\sin 2\theta]$$
$$= r^2\cos 2\theta + i r^2 \sin 2\theta$$

$\therefore u = r^2\cos 2\theta$ & $v = r^2\sin 2\theta$ in the polar form.

2) $f(z) = \log z$.

$$u + iv = \log(re^{i\theta}) = \log r + i\theta \log e$$
$$= \log r + i\theta$$

$u = \log r$ & $v = \theta$ in the polar form.

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

$u = \log\sqrt{x^2+y^2}$, $v = \tan^{-1}(y/x)$ in the cartesian form.

Limit:- A complex valued function $f(z)$ defined in the nhd of a point z_0 is said to have a limit l as z tends to z_0. if for every $\varepsilon > 0$ however small there exists a positive real no δ such that $|f(z) - l| < \varepsilon$ when $|z - z_0| < \delta$. This is written as $\lim_{z \to z_0} f(z) = l$.

Continuity:- A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$

i.e, $|f(z) - f(z_0)| < \varepsilon$ when $|z - z_0| < \delta$

Differentiability :-

A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists & is unique. This limit when exists is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$

Suppose we write $\delta z = z - z_0$ then $z \to z_0$ implies that $\delta z \to 0$.

Hence, $f'(z_0) = \lim\limits_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$

Further $f(z)$ is said to be continuous / differentiable in a domain (or) a region D if $f(z)$ is continuous / differentiable at every point of D.

Analytic Functions :-

A complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if $\frac{dw}{dz} = f'(z) = \lim\limits_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$ exists and is unique at z_0 and in the nhd of z_0

Further $f(z)$ is said to be analytic in a region if it is analytic at every point of the region.

Analytic function is also called a regular function (or) holomorphic function.

We can as well say that $f(z)$ is analytic at a point z_0 if it is differentiable at z_0 & in the nhd of z_0.

Cauchy-Riemann Equations in the Cartesian form

The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that, there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are known as Cauchy-Riemann (C-R) Equations: $u_x = v_y$ and $v_x = -u_y$.

proof:-

Let $f(z)$ be analytic at a point $z = x+iy$ and hence by the definition,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exists \& is unique.}$$

In the cartesian form $f(z) = u(x,y) + iv(x,y)$ and let δz be the increment in z corresponding to the increments $\delta x, \delta y$ in x, y.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + i v(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \left[\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z}\right]$$

$$+ i \lim_{\delta z \to 0} \left[\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta z}\right] \longrightarrow (1)$$

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$\therefore$ Now, $\delta z = (z+\delta z) - z$ where $z = x+iy$

$\delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

$\delta z = \delta x + i\delta y$

$\because$ δz tends to zero we have the following two possibilities.

Case (i) :- Let $\delta y = 0$ So that $\delta z = \delta x$ & $\delta z \to 0$ imply $\delta x \to 0$.

Now (1) becomes.

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

These limits from the basic definition are the partial derivatives of u and v w.r.t x

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (2)$$

Case (ii) :- Let $\delta x = 0$ so that $\delta z = i\delta y$ and $\delta z \to 0$ imply $i\delta y \to 0$ (or) $\delta y \to 0$

Now (1) becomes

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

But $1/i = i/i^2 = i/-1 = -i$ & hence we have,

$$f'(z) = \lim_{\delta y \to 0} -i\,\frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \longrightarrow (3)$$

Equating the RHS of (2) & (3) we have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Now Equating the real & imaginary parts we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus we have established Cauchy-Riemann Equations $u_x = v_y$ and $v_x = -u_y$

These are the necessary conditions in the cartesian form for the complex valued function $f(z) = u + iv$ to be analytic.

Cauchy-Riemann Equations in the polar form

If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ & satisfy the equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

These are known as Cauchy-Riemann (C-R) Equations in the polar form.

Proof:-

Let $f(z)$ be analytic at a point $z = re^{i\theta}$ and hence by definition,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists & is unique.

In the polar form $f(z) = u(r,\theta) + iv(r,\theta)$ and let δz be the increment in z corresponding to the increments $\delta r, \delta\theta$ in r, θ.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + i \lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \longrightarrow (1)$$

Consider $z = re^{i\theta}$. $\because$ z is a function of two variables r, θ.

We have

$$\delta z = \frac{\partial z}{\partial r}\,\delta r + \frac{\partial z}{\partial \theta}\,\delta\theta$$

$$= \frac{\partial}{\partial r}(re^{i\theta})\,\delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\,\delta\theta$$

$$= e^{i\theta}\,\delta r + ire^{i\theta}\,\delta\theta$$

$\therefore$ δz tends to zero, we have the following two possibilites.

Case (i):- Let $\delta\theta = 0$ so that $\delta z = e^{i\theta}\delta r$ & $\delta z \to 0$ imply $\delta r \to 0$

(1) becomes,

$$f'(z) = \lim_{\delta r \to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\delta r}$$

$$+ i \lim_{\delta r \to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}\delta r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \longrightarrow (2)$$

Case (ii) :- Let $\delta r = 0$ so that $\delta z = ire^{i\theta}\delta\theta$ & $\delta z \to 0$ imply $\delta\theta \to 0$

(1) becomes

$$f'(z) = \lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\delta\theta}$$

$$+ i \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{ire^{i\theta}\delta\theta}$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i \lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right]$$

$$= \frac{1}{re^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

But $1/i = i/i^2 = i/-1 = -i$ & hence we have,

$$f'(z) = \frac{1}{re^{i\theta}}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

$$= e^{-i\theta}\left[\frac{-i}{r}\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial v}{\partial\theta}\right].$$

$$= e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial\theta} - \frac{i}{r}\frac{\partial u}{\partial\theta}\right] \longrightarrow (3)$$

Equating the RHS of (2) & (3) we have,

$$e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]=e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{i}{r}\frac{\partial u}{\partial \theta}\right]$$

Cancelling $e^{-i\theta}$ on Both sides and equating the real and imaginary parts we get,

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}=\frac{-1}{r}\frac{\partial u}{\partial \theta}$$

(or) $\quad r u_r = v_\theta \quad$ and $\quad r v_r = -u_\theta$

Thus we have established Cauchy-Riemann Equations in the polar form.

Properties of Analytic Functions :-

1) Harmonic Function :-

A function ϕ is said to be harmonic if it satisfies Laplace's equation $\nabla^2\phi = 0$

In Cartesian form, $\phi(x,y)$ is harmonic if $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$

In polar form, $\phi(r,\theta)$ is harmonic if $\dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial \phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial \theta^2}=0$

~~Haromic p~~ Harmonic property

The real and imaginary parts of an analytic function are harmonic.

Proof :-

we shall prove the result separately for cartesian & polar form of z.

Cartesian form :-

Let $f(z) = u(x,y) + iv(x,y)$ be analytic.
We shall S.T u & v satisfy Laplace's eq^n in the Cartesian form.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\because f(z)$ is analytic we have C-R Equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow (1) \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \longrightarrow (2)$$

Differentiating (1) w.r.t. x & (2) w.r.t. y partially we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

But $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ is always true.

& hence we have,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad (\text{or}) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow u$ is harmonic.

Again differentiating (1) w.r.t y and (2) w.r.t. x partially we get,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$$

But

$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is always true & hence we have

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \quad (\text{or}) \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow v$ is harmonic

Thus we have proved that the real & imaginary parts of an analytic function when expressed in the Cartesian form satisfy Laplace's eq^n in the Cartesian form.

Polar form:-

Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic we shall S.T u & v satisfy Laplace's eq^n in the polar form.

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

we have C-R eq^n in the polar form.

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \longrightarrow (1) \qquad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} \longrightarrow (2)$$

Differentiating (1) w.r.t. r & (2) w.r.t. θ partially we get,

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\cdot 1 = \frac{\partial^2 v}{\partial r \partial \theta} \; ; \; r\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{\partial^2 u}{\partial \theta^2}$$

But

$\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$ is always true & hence we have

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

Dividing by r & transposing the term in the RHS to LHS we obtain:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\therefore$ u satisfies Laplace's eq^n in the polar form $\Rightarrow$ u is harmonic.

Again differentiating (1) w.r.t. θ and (2) w.r.t. r partially we get

$$r\frac{\partial^2 u}{\partial\theta\partial r} = \frac{\partial^2 v}{\partial\theta^2}; \quad r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\cdot 1 = -\frac{\partial^2 u}{\partial r\partial\theta}$$

But

$$\frac{\partial^2 u}{\partial\theta\partial r} = \frac{\partial^2 u}{\partial r\partial\theta}$$ is always true & we have

$$\frac{1}{r}\left[\frac{\partial^2 v}{\partial\theta^2}\right] = -\left[r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\right]$$

Dividing by r & transposing terms in the RHS to LHS we obtain

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0$$

$\therefore$ v satisfies Laplace's eq^n in the polar form

$\Rightarrow v$ is harmonic.

Thus we have proved that u & v are harmonic.

Note:- The converse of this theorem is not true. That is to say that we can give examples of function u & v satisfying Laplace's eq^n but not satisfying C-R equations.

Example:-

Let $u = x^2 - y^2$, $v = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2 \qquad \frac{\partial^2 v}{\partial x^2} = 6x, \quad \frac{\partial^2 v}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x - 6x = 0$$

This shows that u & v are harmonic functions

But C-R eq^n $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

are not satisfied.

Hence $u+iv$ is not analytic.

2) **Orthogonal property :-**

If $f(z) = u+iv$ is analytic then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$, c_1 & c_2 being constants, intersects each other orthogonally.

proof :-

W.K.T two curves intersect each other orthogonally if the tangents at the point of intersection are at right angles. Further w.k.T $\frac{dy}{dx}$ represents slope of the tangent & the condition for perpendicularity of two lines is that the product of their slopes must be equal to -1.

Consider $u(x,y) = c_1$ & differentiating w.r.t. x treating y as a function of x we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

(or) $$\frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1$$

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$$v(x,y) = c_2$$

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = m_2$$

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$$\therefore m_1 m_2 = \frac{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} \longrightarrow (1)$$

But $f(z) = u + iv$ is analytic & hence we have C-R Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Using these in (1) we have,

$$m_1 m_2 = \frac{\frac{\partial v}{\partial y} \cdot \frac{-\partial u}{\partial y}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} = -1$$

Hence the curves intersect orthogonally at every point of intersection.

Note 1:- Converse of this theorem is not true i.e, The curves $u = c_1$ & $v = c_2$ intersect orthogonally but u & v does not satisfy C-R equations.

Note 2:- The result can also be established for the polar family of curves.

If $r = f(\theta)$ w.k.T $\tan\phi = r \cdot \frac{d\theta}{dr}$, ϕ being the angle b/n the radius vector & the tangent.
The angle b/n the tangents at the point of intersection of the curves is $\phi_1 - \phi_2$ & $\tan\phi_1 \cdot \tan\phi_2 = -1$ & the condition for orthogonality.

Consider, $u(r,\theta) = c_1$ & differentiate w.r.t. θ treating r as a function of θ.

$\therefore u_r \dfrac{dr}{d\theta} + u_\theta = 0$ (or) $\dfrac{dr}{d\theta} = -\dfrac{u_\theta}{u_r}$

Hence $\tan\phi_1 = r\dfrac{d\theta}{dr} = -\dfrac{r u_r}{u_\theta}$

1114^{ly} for the curve $v(r,\theta) = c_2$

$$\tan\phi_2 = -\frac{r v_r}{v_\theta}$$

$$\therefore \tan\phi_1 . \tan\phi_2 = \frac{(r u_r)(r v_r)}{u_\theta . v_\theta}$$

But $r u_r = v_\theta$ & $r v_r = -u_\theta$ by C-R equations.

Now $\tan\phi_1 . \tan\phi_2 = \dfrac{(v_\theta)(-u_\theta)}{u_\theta . v_\theta} = -1$

Thus the polar family of curves $u(r,\theta) = c_1$ & $v(r,\theta) = c_2$ intersect each other orthogonally.

Example:- Let $u = \dfrac{x^2}{y}$ and $v = x^2 + 2y^2$

we shall S.T the curves $u = c_1$ & $v = c_2$ intersect orthogonally but u & v does not satisfy C-R eq^n's.

consider, $\dfrac{x^2}{y} = c_1$ & $x^2 + 2y^2 = c_2$

Differentiating these w.r.t. x treating y as a function of x, we obtain

$$\frac{y(2x) - x^2 \frac{dy}{dx}}{y^2} = 0 \ ; \ 2x + 4y\frac{dy}{dx} = 0$$

i.e, $2xy - x^2 \dfrac{dy}{dx} = 0$; $4y \dfrac{dy}{dx} = -2x$

$\dfrac{dy}{dx} = \dfrac{2xy}{x^2} = \dfrac{2y}{x} = m_1$; $\dfrac{dy}{dx} = -\dfrac{x}{2y} = m_2$

Now $m_1 \cdot m_2 = \dfrac{2y}{x} \cdot -\dfrac{x}{2y} = -1$

Hence $u = c_1$ & $v = c_2$ intersect orthogonally.

Further we have,

$\dfrac{\partial u}{\partial x} = \dfrac{2x}{y}$; $\dfrac{\partial u}{\partial y} = -\dfrac{x^2}{y^2}$, $\dfrac{\partial v}{\partial x} = 2x$, $\dfrac{\partial v}{\partial y} = 4y$

C-R eq^{n}s : $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

are not satisfied. Thus we conclude that $u+iv$ is not analytic.

Type 1 :- Finding the derivative of an analytic function

1) Given $w = f(z)$, we substitute $z = x+iy$ (or) $z = re^{i\theta}$ to find the real & imaginary parts u & v as functions of x, y (or) r, θ.

2) We find first order partial derivatives & verify Cauchy-Riemann eq^{n}s in the Cartesian (or) polar form to conclude that $f(z)$ is analytic

3) To find the derivative of $f(z)$ we make use of the fundamental results derived while establishing C-R eq^{n}s. They are as follows:

$f'(z) = u_x + iv_x$ — cartesian form

$f'(z) = e^{-i\theta}[u_r + iv_r]$ — polar form

4) We substitute for the partial derivatives and re-arrange as a function of $(x+iy)$ (or) $re^{i\theta}$ which is z, with the result $f'(z)$ is obtained as a function of z.

Problems:-

1) Show that $f(z) = z^n$ is analytic. Hence find its derivative

=)

Given $f(z) = z^n$

Taking $z = re^{i\theta}$ we have,

$$u + iv = (re^{i\theta})^n = r^n e^{in\theta}$$

$$u + iv = r^n [\cos n\theta + i \sin n\theta]$$

$\therefore u = r^n \cos n\theta$ & $v = r^n \sin n\theta$.

$u_r = nr^{n-1} \cos n\theta$ $\quad v_r = nr^{n-1} \sin n\theta$

$u_\theta = -nr^n \sin n\theta$ $\quad v_\theta = nr^n \cos n\theta$

C-R equations in the polar form.

$ru_r = v_\theta$ $\quad rv_r = -u_\theta$

$r . nr^{n-1} \cos n\theta$ $\quad rnr^{n-1} \sin n\theta = -nr^n \sin n\theta$

$= nr^n \cos n\theta$ $\quad nr^n \sin n\theta = -nr^n \sin n\theta$

$nr^n \cos n\theta = nr^n \cos n\theta$

C.R eq^n's are satisfied.

Thus $f(z) = z^n$ is analytic

Also we have,

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

$$f'(z) = e^{-i\theta}[nr^{n-1} \cos n\theta + inr^{n-1} \sin n\theta)$$

$$= nr^{n-1} e^{-i\theta}(\cos n\theta + i \sin n\theta)$$

$$= nr^{n-1} e^{-i\theta} . e^{in\theta}$$

$$= nr^{n-1} e^{i\theta(n-1)}$$

$$= nr^{n-1} [e^{i\theta}]^{(n-1)}$$

$$= n(re^{i\theta})^{n-1}$$

$$f'(z) = nz^{n-1}$$

2) Show that $w = z + e^z$ is analytic & Hence find $\frac{dw}{dz}$

$\Rightarrow$ Given: $w = z + e^z$

$u + iv = (x+iy) + e^{(x+iy)}$

$= (x+iy) + e^x . e^{iy}$

$= (x+iy) + e^x [\cos y + i\sin y]$

$u + iv = (x + e^x\cos y) + i(y + e^x \sin y)$

$\therefore u = x + e^x \cos y \qquad v = y + e^x \sin y$

$u_x = 1 + e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = 1 + e^x \cos y$

C-R equations in cartesian form is satisfied

$u_x = v_y$ & $v_x = -u_y$ are satisfied.

$\therefore w = z + e^z$ is analytic.

Also we have $\frac{dw}{dz} = f'(z) = u_x + i v_x$.

$f'(z) = (1 + e^x \cos y) + i(e^x \sin y)$

$= 1 + e^x(\cos y + i\sin y)$

$= 1 + e^x . e^{iy}$

$= 1 + e^{x+iy}$

$f'(z) = 1 + e^z, \quad z = x+iy$

3) Show that the function $f(z) = \sin 2z$ is analytic. & hence find its derivative.

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$\Rightarrow$ Given $f(z) = \sin 2z$

$$u + iv = \sin 2(x+iy)$$
$$= \sin 2x \cos 2iy + \cos 2x \sin 2iy$$
$$= \sin 2x \cos 2hy + \cos 2x (i \sin 2hy)$$
$$= \sin 2x \cos 2hy + i \cos 2x \sin 2hy$$

$u = \sin 2x \cos 2hy$, $v = \cos 2x \sin 2hy$

$u_x = 2\cos 2x \cos 2hy$ $\quad$ $v_x = -2\sin 2x \sin 2hy$

$u_y = 2\sin 2x \sin 2hy$ $\quad$ $v_y = 2\cos 2x \cos 2hy$

$\therefore$ C-R equation $u_x = v_y$ & $v_x = -u_y$ are satisfied.

Thus $f(z) = \sin 2z$ is analytic.

we have,

$$f'(z) = u_x + i v_x$$
$$= 2\cos 2x \cos 2hy + i(-2\sin 2x \sin 2hy)$$
$$= 2\cos 2x \cos 2hy - i 2\sin 2x \sin 2hy$$

Using $\cosh 2y = \cos 2iy$ & $i \sinh 2y = \sin 2iy$

$$f'(z) = 2\cos 2x \cos 2iy - 2\sin 2x \sin 2iy$$
$$= 2\cos(2x + i2y)$$
$$= 2\cos 2(x+iy)$$
$$f'(z) = 2\cos 2z$$

4) S.T $w = \log z$, $z \neq 0$ is analytic find $\frac{dw}{dz}$

$\Rightarrow$

$w = \log z$

$w = \log(re^{i\theta})$ $\quad$ $\because z = re^{i\theta}$

$u + iv = \log r + \log(e^{i\theta})$

$= \log r + i\theta$ $\quad$ $\because \log_e e = 1$

$u = \log r$, $v = \theta$

$u_r = \frac{1}{r}$ $v_r = 0$

$u_\theta = 0$ $v_\theta = 1$

C-R eqⁿ in polar form

$r u_r = u_\theta$ & $r v_r = -u_\theta$ are satisfied

Thus $w = \log z$ is analytic

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\frac{1}{r} + i.0\right)$$

$$= \frac{1}{re^{i\theta}}$$

$$f'(z) = \frac{1}{z}$$

Finding the conjugate harmonic function and the analytic function.

We have proved that the real & imaginary parts of an analytic function $f(z) = u + iv$ are harmonic.
u & v are called conjugate harmonic functions.
Given u we can find v & vice-versa.

procedure:-

1) Given u, we find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$.

2) We consider C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

3) Substituting for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ we obtain a system of two non-homogeneous PDE of the form $\frac{\partial v}{\partial y} = f(x,y)$; $\frac{\partial v}{\partial x} = g(x,y)$

4) These can be solved by direct integration to obtain the required v.

5) The same procedure is adopted to find u given v.

6) Further u+iv will give us $f(z)$ as a function of x, y.

putting $x = z$, $y = 0$ we can obtain $f(z)$ we can obtain $f(z)$ as a function of z.

problems:

1) Show that $u = e^x(x\cos y - y\sin y)$ is harmonic & find its harmonic conjugate.

=)

$$u = e^x(x\cos y - y\sin y)$$

$$u_x = e^x\cos y + (x\cos y - y\sin y)e^x$$

$$\rightarrow u_x = e^x[\cos y + x\cos y - y\sin y)$$

$$u_{xx} = e^x \cdot \cos y + (\cos y + x\cos y - y\sin y)e^x$$

$$\rightarrow u_{xx} = e^x[2\cos y + x\cos y - y\sin y) \longrightarrow (1)$$

Also $u_y = e^x(-x\sin y - [y\cos y + \sin y])$

$$u_y = -e^x[x\sin y + y\cos y + \sin y)$$

$$u_{yy} = -e^x\{x\cos y + [-y\sin y + \cos y] + \cos y\}$$

$$u_{yy} = -e^x[2\cos y + x\cos y - y\sin y) \longrightarrow (2)$$

(1) + (2) gives $u_{xx} + u_{yy} = 0$

$\therefore u$ is harmonic

Now C-R eqⁿ's $u_x = v_y$ & $v_x = -u_y$

$\Rightarrow \quad v_y = e^x(\cos y + x\cos y - y\sin y) \longrightarrow (3)$

$v_x = e^x(x\sin y + y\cos y + \sin y) \longrightarrow (4)$

from (3)

$$v = e^x\left[\int \cos y\,dy + x\int \cos y\,dy - \int y\sin y\,dy\right] + f(x)$$

$$v = e^x\left[\sin y + x\sin y - \left[y.(-\cos y) + \int \cos y\,dy\right]\right] + f(x)$$

$$v = e^x\left[\sin y + x\sin y + y\cos y - \sin y\right] + f(x)$$

$$v = e^x\left[x\sin y + y\cos y\right] + f(x) \longrightarrow (5)$$

from (4)

$$v = \sin y\int xe^x dx + y\cos y\int e^x dx + \sin y\int e^x dx + g(y)$$

$$v = \sin y\,(xe^x - e^x) + y\cos y.e^x + \sin y\,e^x + g(y)$$

$$v = xe^x\sin y + ye^x\cos y + g(y) \longrightarrow (6)$$

Comparing (5) & (6) we must choose

$f(x) = 0 \quad , \quad g(y) = 0$

$\therefore \quad v = xe^x\sin y + e^x y\cos y$

$v = e^x[x\sin y + y\cos y]$

Now $f(z) = u + iv$

$f(z) = e^x[x\cos y - y\sin y] + ie^x[x\sin y + y\cos y]$

putting $x = z$, $y = 0$

we get $f(z) = ze^z$

2) Show that $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic.

find its harmonic conjugate & also corresponding analytic function.

⇒

$$u = \left(r + \frac{1}{r}\right)\cos\theta.$$

We shall S.T $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \longrightarrow (1)$

$$u_r = \left\{\left(1 - \frac{1}{r^2}\right)\cos\theta\right\} \qquad u_{rr} = \frac{2\cos\theta}{r^3}$$

$$u_\theta = -\left(r + \frac{1}{r}\right)\sin\theta \qquad u_{\theta\theta} = -\left(r + \frac{1}{r}\right)\cos\theta$$

LHS of (1) ⇒

$$\frac{2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta = 0$$

∴ u is harmonic.

To find v, let us consider C-R eq^{ns} in the polar form

$$r u_r = v_\theta \quad \& \quad ; \; r v_r = -u_\theta$$

$$v_\theta = \left(r - \frac{1}{r}\right)\cos\theta \quad ; \quad v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta$$

Int Int

$$v = \left(r - \frac{1}{r}\right)\sin\theta + f(r) \quad ; \quad v = \left(r - \frac{1}{r}\right)\sin\theta + g(\theta)$$

comparing we must have $f(r) = 0$, $g(\theta) = 0$

∴ The required harmonic conjugate $v = \left(r - \frac{1}{r}\right)\sin\theta$

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Also, $f(z) = u + iv$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$\cancel{= r\cos\theta +}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= re^{i\theta} + \frac{1}{re^{+i\theta}}$$

$f(z) = z + \frac{1}{z}$ is the analytic function

(or) put $r = z$ & $\theta = 0$

$$f(z) = z + \frac{1}{z}$$

3) Given that $u = x^2 + 4x - y^2 + 2y$ as the real part of an analytic function find v and hence find $f(z)$ in terms of z.

=)

$$u = x^2 + 4x - y^2 + 2y$$

$u_x = 2x + 4 \qquad u_y = -2y + 2$

C-R eq^{n}s $\Rightarrow$ $u_x = v_y$ & $v_x = -u_y$

$v_y = 2x + 4 \qquad v_x = 2y - 2$

Int $\qquad\qquad$ Int

$v = 2xy + 4y + f(x)$, $\quad v = 2xy - 2x + g(y)$

comparing, we choose $f(x) = -2x$ & $g(y) = 4y$

$\therefore v = 2xy + 4y - 2x$

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$\therefore f(z) = u + iv$

$= (x^2 + 4x - y^2 + 2y) + i(2xy + 4y - 2x)$

put $x = z$ & $y = 0$

$\therefore f(z) = z^2 + 4z - 2iz$

* Construction of analytic function $f(z)$ given its real (or) imaginary part:-

1) Given u (or) v as function of x, y we find u_x, u_y (or) v_x, v_y & consider

$f'(z) = u_x + i v_x$.

2) Given u, we use C-R eq^n $v_x = -u_y$ (or) given v we use C-R eq^n $u_x = v_y$ so that

$f'(z) = u_x - i u_y$ (or) $f'(z) = v_y + i v_x$

3) we substitute the expression for the partial derivatives in the RHS & then put $x = z$ & $y = 0$ to obtain $f'(z)$ as a function of z.

4) Integrating w.r.t. z we get $f(z)$.

5) 111^{ly} In the case of polar co-ordinates r, θ we consider

$f'(z) = e^{-i\theta}(u_r + i v_r)$ & use C-R eq^n in RHS

$v_r = \frac{-1}{r} u_\theta$ given u (or) $u_r = \frac{1}{r} v_\theta$ given v.

6) we use the substitution $r = z$ & $\theta = 0$ to obtain $f'(z)$ as a function of z

7) Integrating w.r.t. z we get $f(z)$

This Method is known as Milne's thomson Method.

problems :-

1) find the analytic function $f(z) = u + iv$, where $u = x^2 - y^2 + \frac{x}{x^2+y^2}$.

=)

$u = x^2 - y^2 + \frac{x}{x^2+y^2}$

$u_x = 2x + \frac{(x^2+y^2 - 2x^2)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + 2x$

$u_y = -2y + \frac{(x^2+y^2).0 - 2xy}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$

$f'(z) = u_x + iv_x$ But In C-R eq^n $v_x = -u_y$

$f'(z) = u_x - iu_y$

$f'(z) = \left(\frac{y^2 - x^2}{(x^2+y^2)^2} + 2x\right) + i\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)$

put $x = z$ & $y = 0$.

$f'(z) = \frac{-z^2}{z^4} + 2z$

$f'(z) = 2z - \frac{1}{z^2}$

Int

$f(z) = z^2 + \frac{1}{z} + c$

2) find the analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ where, $u(r,\theta) = r^2 \cos 2\theta$

$\Rightarrow$ $u = r^2 \cos 2\theta$

$u_r = 2r\cos 2\theta \quad , \quad u_\theta = -2r^2 \sin 2\theta$

$f'(z) = e^{-i\theta}(u_r + i v_r)$

But $v_r = -\frac{1}{r} u_\theta$ (C-R eq^n)

$f'(z) = e^{-i\theta}\left(u_r - \frac{i}{r} u_\theta\right)$

$f'(z) = e^{-i\theta}\left[2r\cos 2\theta - \frac{i}{r}(-2r^{\cancel{2}}\sin 2\theta)\right]$

$= e^{-i\theta}[2r\cos 2\theta + i\, 2r\sin 2\theta)$

$= 2r\, e^{-i\theta}[\cos 2\theta + i\sin 2\theta) \longrightarrow (1)$

$= \cancel{2r}e^{-}$ put $r = z$, $\theta = 0$

$f'(z) = 2z$

Int

$f(z) = z^2 + c$

3) find the analytic function

$f(z) = u(r,\theta) + i v(r,\theta)$ where

$v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$

$\Rightarrow$

$v = r^2 \cos 2\theta - r\cos\theta + 2$

$v_r = 2r\cos 2\theta - \cos\theta$ $\quad v_\theta = -2r^2 \sin 2\theta + r\sin\theta$

$f'(z) = e^{-i\theta}(u_r + i v_r)$

$u_r = \frac{1}{r} v_\theta$

$f'(z) = e^{-i\theta}\left[\frac{1}{r} v_\theta + i v_r\right]$

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$$f'(z) = e^{-i\theta}\left[(-2r\sin 2\theta + \sin\theta) + i(2r\cos 2\theta - \cos\theta)\right]$$

$$= e^{-i\theta}\left[2r(\cos\right.$$

put $r = z$ & $\theta = 0$

$$f'(z) = (2z\theta - 1)i$$

$$\cancel{= 0}$$

Int

$$f(z) = i\left(2\theta \cdot \frac{z^2}{2} - z\right) + c$$

$$f(z) = i(z^2 - z) + c$$

$$f(z) = iz(\theta z - 1) + c$$

Miscellaneous problems :-

1) If $f(z)$ is analytic, S.T $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$

=)

let $f(z) = u + iv$ be analytic

$\therefore |f(z)| = \sqrt{u^2 + v^2}$

(or) $|f(z)|^2 = u^2 + v^2 = \phi$ (say)

To p.T $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

i.e, $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

consider,

$\phi = u^2 + v^2$ & diff w.r.t. x partially

$$\phi_x = 2uu_x + 2vv_x$$

$$= 2[uu_x + vv_x]$$

Diff w.r.t. x again

$$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2] \longrightarrow (1)$$

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$$\phi_{yy} = 2\left[u u_{yy} + u_y^2 + v v_{yy} + v_y^2\right] \longrightarrow (2)$$

Add (1) & (2)

$$\phi_{xx} + \phi_{yy} = 2\left[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2\right] \longrightarrow (3)$$

$\because$ $f(z) = u + iv$ is analytic, u & v are harmonic

Hence $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

Further we also have C-R eq's: $v_y = u_x$ & $u_y = -v_x$

use these results in the RHS of (3)

$\Rightarrow$

$$\phi_{xx} + \phi_{yy} = 2\left[u.0 + v.0 + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right]$$

$$= 2\left[2u_x^2 + 2v_x^2\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

But $f'(z) = u_x + i v_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2}$ (or) $|f'(z)|^2 = u_x^2 + v_x^2$

$$\therefore \phi_{xx} + \phi_{yy} = 4|f'(z)|^2$$

2) If $f(z) = u + iv$ is analytic find f if $u - v = (x-y)(x^2 + 4xy + y^2)$

$\Rightarrow$ $u - v = x^3 + 3x^2y - 3xy^2 - y^3$

$u_x - v_x = 3x^2 + 6xy - 3y^2 \longrightarrow (1)$

$u_y - v_y = 3x^2 - 6xy - 3y^2$

But $u_y = -v_x$ & $v_y = u_x$ by C-R eq^{n}s

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$$-V_x - u_x = 3x^2 - 6xy - 3y^2 \longrightarrow (2)$$

Solve (1) & (2)

(1) + (2)

$$u_x - V_x = 3x^2 + 6xy - 3y^2$$

$$\underline{-u_x - V_x = 3x^2 - 6xy - 3y^2}$$

$$-2V_x = 6x^2 - 6y^2$$

$$V_x = 3(y^2 - x^2)$$

(1) − (2)

$$u_x - V_x = 3x^2 + 6xy - 3y^2$$

$$\underline{u_x + V_x = -3x^2 + 6xy + 3y^2}$$

$$2u_x = 12xy$$

$$u_x = 6xy$$

$$f'(z) = u_x + iV_x$$

$$= 6xy + i.\,3(y^2 - x^2)$$

put $x = z,\ y = 0$

$$f'(z) = -3iz^2$$

Int.

$$f(z) = -iz^3 + c$$

3) If $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic & given that $u + V = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$, $r \neq 0$ determine the analytic function $f(z)$

=)

$$u + V = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$$

D.w.r.t. 'r' & also 'θ' ~~par~~

$$U_r + V_r = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) \longrightarrow (1)$$

$$u_\theta + V_\theta = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$$

By C-R eq^{n}s $V_\theta = r u_r$ & $-u_\theta = r V_r$

$$-rV_r + r u_r = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$$

$$u_r - V_r = \frac{-2}{r^3}(\sin 2\theta + \cos 2\theta) \longrightarrow (2)$$

Solve (1) & (2)

$$(1)+(2) \Rightarrow 2u_r = \frac{-4\cos 2\theta}{r^3}$$

$$(\text{or}) \quad u_r = \frac{-2}{r^3}\cos 2\theta$$

$$(1)-(2) \Rightarrow 2V_r = \frac{4}{r^3}\sin 2\theta \quad (\text{or}) \quad V_r = \frac{2}{r^3}\sin 2\theta$$

$$f'(z) = e^{-i\theta}(u_r + iV_r)$$

$$= e^{-i\theta}\left[\frac{-2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right]$$

$$= \frac{-2}{r^3} e^{-i\theta}\left[\cos 2\theta - i\sin 2\theta\right]$$

$$= \frac{-2}{r^3} e^{-i\theta} \cdot e^{-2i\theta}$$

$$= \frac{-2}{r^3} e^{-3i\theta} = \frac{-2}{(re^{i\theta})^3} = \frac{-2}{z^3}$$

$$f'(z) = \frac{-2}{z^3}$$

Int

$$f(z) = \frac{1}{z^2} + C$$