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UNIT - 3

Time Domain Representation of LTI System

Solution of difference equation

A discrete time system can be represented by a constant coefficent difference eqn as.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \longrightarrow (1)$$

Here $x(n-k)$ are i/ps and $y(n-k)$ are ops.

Equation (1) can be written as.

$$a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = -\frac{1}{a_0} \sum_{k=1}^{M} a_k y(n-k) + \frac{1}{a_0} \sum_{k=0}^{M} b_k x(n-k)$$

The difference equation can be solved to obtain an expression for o/p $y(n)$. This expression for $y(n)$ is made up of two components.

1. Natural response (zero i/p response)
2. Forced response (zero state response)

1) Natural response is the o/p of the system with zero i/p.
This response is obtained only with initial conditions. It is denoted by $y^{(n)}(n)$.

2) Forced response is the o/p of the system for a given i/p with zero initial conditions.
It is denoted as $y^{(f)}(n)$.

The complete response of the system is then given by the sum of natural response and the forced response.

Natural response

It is known that natural response is obtained for zero inputs.

$\therefore$ equation ① $\Rightarrow$

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \longrightarrow ②$$

equation ② is called as homogenous diff equation.

writing the characteristic equation

$$\sum_{k=0}^{N} a_k \gamma^{N-k} = 0$$

$$a_0 \gamma^N + a_1 \gamma^{N-1} + \cdots\cdots + a_N = 0 \longrightarrow ③$$

Form of natural response

i) If roots are real and distinct.

$$y^{(n)}(n) = c_1 \gamma_1^n + c_2 \gamma_2^n + \cdots + c_N \gamma_N^n$$

ii) If roots are complex conjugate (imagenary)

i.e $\gamma e^{j\Omega}$

$$y^{(n)}(n) = \gamma^n [c_1 \cos n\Omega + c_2 \sin n\Omega]$$

ii) If roots are real and repeated.

$$y^{(n)}(n) = \gamma^n [c_0 + c_1 n + c_2 n^2 + \cdots + c_p n^p]$$

Steps to find the natural response

① write the homogenous equation.

② write the characteristic equation and find its root.

③ Write the form of natural response.

④ Evaluate the constants using initial conditions

⑤ Substitute constants in form of natural response.

1) Determine the natural response (zero input response) for the following system.

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + x(n-1) \quad \text{with}$$

$y(-1) = 0$, $y(-2) = 1$.

Sol: Step 1: Write the homogenous equation.

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 0 \longrightarrow ①$$

order $N = 2$.

Step 2: The characteristic equation and find its roots.

$$\sum_{k=0}^{N} a_k r^{N-k}, \quad \sum_{k=0}^{2} a_k r^{2-k} = 0$$

$$a_0 r^2 + a_1 r^1 + a_2 = 0 \longrightarrow ②$$

compare ① and ②.

$a_0 = 1$, $a_1 = -\frac{1}{4}$, $a_2 = -1/8$

② ⇒ $r^2 - \frac{1}{4}r - \frac{1}{8} = 0$

roots. $r_1 = 1/2$ and $r_2 = -1/4$.

Step 3: Form of natural response:

roots are real and distinct

$$\therefore y^{(n)}(n) = c_1 r_1^n + c_2 r_2^n + \dots + c_N r_N^n$$

$N = 2$, $r_1 = 1/2$, $r_2 = -1/4$.

$$\therefore y^{(n)}(n) = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(-\tfrac{1}{4}\right)^n \longrightarrow ③$$

Step 4: Evaluate the constants using initial conditions.

Put $n = 0$ in eq ③

$y^{(n)}(0) = c_1(1/2)^0 + c_2(-1/4)^0$

$y^{(n)}(0) = c_1(1) + c_2(1) = c_1 + c_2$ — (4)

Put $n=1$ in eq (3)

$y^{(n)}(1) = c_1(\frac{1}{2})^1 + c_2(-1/4)^1$

$y^{(n)}(1) = \frac{1}{2}c_1 - \frac{1}{4}c_2$ — (5)

Now put $n=0$ in eq (1)

$y(0) - \frac{1}{4}y(0-1) - \frac{1}{8}y(0-2) = 0$

$y(0) = \frac{1}{4}y(-1) + \frac{1}{8}y(-2)$

substitute $y(-1) = 0$, $y(-2) = 1$ (Given)

$\therefore y(0) = \frac{1}{4}\times 0 + \frac{1}{8}\times 1 = 0 + \frac{1}{8} = 1/8$

Substitute $y(0) = 1/8$ in eq (4)

$4 \Rightarrow c_1 + c_2 = 1/8 \longrightarrow$ (6)

Now put $n=1$ in eq (1)

$y(1) - \frac{1}{4}y(1-1) - \frac{1}{8}y(1-2) = 0$

$y(1) = \frac{1}{4}y(0) + \frac{1}{8}y(-1)$

$y(1) = \frac{1}{4}\times 1/8 + 1/8 \times 0$

$y(1) = \frac{1}{32}$

substitute $y(1) = 1/24$ in eq (5)

(5) $\Rightarrow \frac{1}{2}c_1 - \frac{1}{4}c_2 = 1/32. \longrightarrow$ (7)

Solving (6) and (7)

$c_1 = 1/12$ and $c_2 = 1/24.$

Step 5 : substitute the constants in form of natural response.

$\therefore y^{(n)}(n) = \frac{1}{12}(1/2)^n + 1/24(-1/4)^n$

Forced Response

Forced response is the sum of the two components. $y^{(f)}(n) = y^{(n)}(n) + y^{(p)}(n)$.

$y^{(f)}(n)$ = Natural response + particular ~~integral~~ solution.

The particular solution is solution of difference Equation for given input.

The particular solution has the same form as that of i/p.

SL No	input $x(n)$	Particular Solution.
1.	1	K
2.	* α^n	$K\alpha^n$
3.	$\cos(\Omega n+\phi)$	$K_1\cos\Omega n + K_2\sin\Omega n$
4.	* $\alpha^n\cos(\Omega n+\phi)$	$\alpha^n[K_1\cos\Omega n + K_2\sin\Omega n]$
5.	n	$K_0 + K_1 n$.
6.	n^P	$K_0 + K_1 n + K_2 n^2 + \dots + K_P n^P$
7.	* $n\alpha^n$	$\alpha^n(K_0 + K_1 n)$
8.	* $n^P\alpha^n$	$\alpha^n(K_0 + K_1 n + K_2 n^2 + \dots + K_P n^P)$

* Note: If α is one of the roots of char. eqn repeated m no of times then $y_p(n)$ must be multiplied by n^m.

Steps to find forced response

① write homogenous equation.

② write characteristic equation and find its roots.

③ write the form of natural response.

④ write the form of particular solution.

⑤ Evaluate the values of constants of particular solution using given diff equation $(y(n) \longrightarrow y^{(p)}(n))$

⑥ write form of forced response as.

$$y^{(f)}(n) = y^{(n)}(n) + y^{(p)}(n)$$

⑦ Evaluate the constants of above equation. with zero initial condition.

⑧ Substitute the constants in $y^{(f)}(n)$.

2) Determine the forced response for the follow system $y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + x(n-1)$ for $x(n) = (1/8)^n u(n)$ assuming zero initial condition.

Sol: Step1 : write homogenous equation.

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 0 \longrightarrow ①$$

order $N = 2$

Step 2 : write characterstic eq and find its roots.

$$\sum_{k=0}^{N} a_k \gamma^{N-k} = 0, \quad \sum_{k=0}^{2} a_k \gamma^{N-k} = 0$$

$$a_0\gamma^2 + a_1\gamma^1 + a_2 = 0 \longrightarrow ②$$

Here, $a_0 = 1$, $a_1 = -1/4$, $a_2 = -\frac{1}{8}$.

eq ② $\Rightarrow$ $\gamma^2 - \frac{1}{4}\gamma - \frac{1}{8} = 0$

roots are $\gamma_1 = 1/2$ and $\gamma_2 = -1/4$.

Step ③ : write the form of natural response.

$$y^{(n)}(n) = c_1\gamma_1^n + c_2\gamma_2^n.$$

$$y^{(n)}(n) = c_1(1/2)^n + c_2(-1/4)^n$$

Step ④ : write the form of particular solution.

$$x(n) = (1/8)^n u(n)$$

$$y^{(p)}(n) = K\left(\tfrac{1}{8}\right)^n u(n) \qquad \because x(n) = \alpha^n$$

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Step 5 : Evaluate the constants in particular sol

Replace $y(n)$ by $y^{(p)}(n) = k(1/8)^n u(n)$ in the difference equation.

$$\Rightarrow k(1/8)^n u(n) - \frac{1}{4}k(1/8)^{n-1}u(n-1) - \frac{1}{8}k(1/8)^{n-2}u(n-2) =$$
$$(1/8)^n u(n) + \left(\frac{1}{8}\right)^{n-1} u(n-1)$$

For $n \geq 2$ all the terms in the above equation will be present. Hence we will obtain value of 'k' for $n \geq 2$. Then we can drop all the unit step function in above equation.

$$k\left(\frac{1}{8}\right)^n - \frac{1}{4}k\left(\frac{1}{8}\right)^{n-1} - \frac{1}{8}k\left(\frac{1}{8}\right)^{n-2} = \left(\frac{1}{8}\right)^n + \left(\frac{1}{8}\right)^{n-1}$$

$$k\left(\frac{1}{8}\right)^2 - \frac{1}{4}\left(\frac{1}{8}\right)^{2-1}k - \frac{1}{8}\left(\frac{1}{8}\right)^{2-2}k = \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^{2-1}$$

$$k\left[\frac{1}{64} - \frac{1}{32} - \frac{1}{8}\right] = \frac{1}{64} + \frac{1}{8}$$

$$k = -1$$

$\therefore$ The particular solution is

$$y^{(p)}(n) = -1\left(\frac{1}{8}\right)^n u(n)$$

Step 6 : Write the form of forced response.

$$y^{(f)}(n) = y^{(p)}(n) + y^{(h)}(n)$$

$$y^{(f)}(n) = c_1(1/2)^n + c_2(-1/4)^n - \left(\frac{1}{8}\right)^n \longrightarrow (3) \quad n \geq 2$$

Step 7 : Find the constants using zero initial condition.

The given system is.

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2) + x(n) + x(n-1) \longrightarrow (4)$$

Put $n=0$ in eq ④

$$y(0) = \frac{1}{4}y(-1) + \frac{1}{8}y(-2) + x(0) + x(-1)$$

In above equation $y(-1) = y(-2) = 0$ (Zero initial condition)

w.k.t $x(n) = \left(\frac{1}{8}\right)^n u(n)$

with $n=0$

$$x(0) = \left(\tfrac{1}{8}\right)^0 u(0) = 1$$

$$x(0) = 1$$

with $n=-1$

$$x(-1) = \left(\frac{1}{8}\right)^{-1} u(-1) = 0$$

$$x(-1) = 0$$

$\therefore$ $y(0) = 0 + 0 + 1 + 0$

$$\boxed{y(0) = 1}$$

Put $n=1$ in eq ④

$$y(1) = \frac{1}{4}y(0) + \frac{1}{8}y(-1) + x(1) + x(0)$$

$$y(1) = \frac{1}{4} + \frac{1}{8} + 1$$

$$= \frac{3}{8} + 1 = \frac{11}{8}$$

$$\boxed{y(1) = \frac{11}{8}}$$

Put $n=0$ in eq ③

$$y^{(f)}(0) = c_1(1/2)^0 + c_2(-1/4)^0 - (1/8)^0$$

$$1 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 2 \rightarrow ⑤$$

Put $n=1$ in eq ③

$$y^{(f)}(1) = c_1(1/2)^1 + c_2(-1/4)^1 - (1/8)^1$$

$$\frac{11}{8} = \frac{1}{2}c_1 - \frac{1}{4}c_2 - \frac{1}{8} \Rightarrow 1/2\, c_1 - 1/4\, c_2 = \frac{12}{8} = \frac{3}{2}.$$

$$\Rightarrow 2c_1 - c_2 = 6 \rightarrow ⑥$$

Solving ⑤ and ⑥ $c_1 = 8/3$ and $c_2 = -2/3$

Step ⑧: substitute the costants in forced response.

$$y^{(f)}(n) = 8/3(1/2)^n - 2/3(-1/4)^n - \left(\frac{1}{8}\right)^n \qquad n \geq 2$$

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Find the forced response of the following difference equations:

i) $y(n) - 7y(n-1) + 12y(n-2) = x(n)$ for $x(n) = n$

ii) $y(n) - y(n-1) = x(n)$ where $x(n) = \cos 2n$

i) Given

$$y(n) - 7y(n-1) + 12y(n-2) = n.$$

I) Homogeneous equation:

$$y(n) - 7y(n-1) + 12y(n-2) = 0$$

Characteristic equation

$$r^2 - 7r + 12 = 0$$

$$r^2 - 4r - 3r + 12 = 0$$

$$r(r-4) - 3(r-4) = 0$$

$$(r-4)(r-3) = 0$$

$$\therefore \boxed{r_1 = 4} \quad \boxed{r_2 = 3}$$

$$y^h(n) = c_1(r_1)^n + c_2(r_2)^n$$

$$y^h(n) = c_1(4)^n + c_2(3)^n$$

Now, $y^p(n) = k_0 + k_1 n.$

Substituting $y^p(n)$ in difference equation,

$$(k_0 + k_1 n) - 7[k_0 + k_1(n-1)] + 12[k_0 + k_1(n-2)] = n$$

$$k_0 + k_1 n - 7[k_0 + k_1 n - k_1] + 12[k_0 + k_1 n - 2k_1] = n$$

$$k_0 + k_1 n - 7k_0 - 7k_1 n + 7k_1 + 12k_0 + \underline{12k_1 n} - 24k_1 = n$$

$$k_0 - 7k_0 + 12k_0 + 7k_1 - 24k_1 + 6k_1 n = n.$$

comparing co-efficients of n on both sides

$$6k_1 = 1$$

$$\boxed{k_1 = 1/6}$$

Comparing the constant terms on both sides, we get

$k_0 - 7k_0 + 12k_0 + 7k_1 - 24k_1 = 0$

$6k_0 - 17k_1 = 0$

$6k_0 - 17\left(\frac{1}{6}\right) = 0$ $\quad (\because k_1 = \frac{1}{6})$

$6k_0 = \frac{17}{6}$

$\boxed{k_0 = \frac{17}{36}}$

$\therefore y_p(n) = \frac{17}{36} + \frac{n}{6}$ $\quad [\because y_p(n) = k_0 + k_1 n]$

Now, Forced response = Natural response + particular solution

$y^f(n) = c_1(4)^n + c_2(3)^n + \frac{n}{6} + \frac{17}{36}$ — ①

Put $n=0$ in eq ①

$y(0) = c_1(4)^0 + c_2(3)^0 + 0 + \frac{17}{36}$

$y(0) = c_1 + c_2 + \frac{17}{36}$ — ②

Put $n=1$ in eq ①

$y(1) = c_1(4) + c_2(3) + \frac{1}{6} + \frac{17}{36}$

$y(1) = 4c_1 + 3c_2 + \frac{23}{36}$ — ③

Put $n=0$ in difference equation

$y(0) - 7y(-1) + 12y(-2) = 0$

$y(0) - 0 + 0 = 0$

$\boxed{y(0) = 0}$

[$\because$ for forced response take initial conditions as zero]

Put $n=1$ in difference equation

$y(1) - 7y(0) + 12y(-1) = 0$

$\boxed{y(1) = 1}$

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Now eq (2) becomes

$$C_1 + 1 C_2 = -\frac{17}{36} \quad -(4)$$

& eq (3) becomes.

$$4C_1 + 3C_2 = -\frac{23}{36} + 1 \quad -(5).$$

Solving (4) & (5)

we get, $C_1 = \frac{16}{9}$

$C_2 = -9/4.$

$$\therefore y^f(n) = \frac{16}{9}(4)^n + (-9/4)(3)^n + \frac{n}{6} + \frac{17}{36}$$

ii) $y(n) - y(n-1) = x(n)$ where $x(n) = \cos 2n$.
(1).

I) Homogeneous equation:

$y(n) - y(n-1) = 0.$

II) Characteristic equation

$r - 1 = 0$

$\Rightarrow \boxed{r = 1}$

$y^h(n) = C u(n)$

For the input $x(n) = \cos 2n$, particular sol^n is of the form

$y^p(n) = k_1 \cos 2n + k_2 \sin 2n.$

Substituting above equation in equation (1).

$$k_1 \cos 2n + k_2 \sin 2n - [k_1 \cos[2(n-1)] + k_2 \sin[2(n-1)]] = \cos 2n.$$

$$k_1 \cos 2n + k_2 \sin 2n - [k_1 [\cos(2n-2)] + k_2 \sin(2n-2)] = \cos 2n$$

$k_1\cos 2n + k_2\sin 2n - [k_1\cos 2n\cos 2 + k_2\sin(2n)\sin 2$

$+ k_2\sin(2n)\cos 2 - k_2\cos(2n)\sin(2)] = \cos 2n$

$[\because \cos(A-B) = \cos A\cos B + \sin A\sin B]$
$\& \sin(A-B) = \sin A\cos B - \cos A\sin B$

$k_1\cos 2n + k_2\sin 2n - k_1\cos 2n\cdot\cos 2 - k_1\sin(2n)\sin 2$
$- k_2\sin(2n)\cos 2 + k_2\cos(2n)\sin 2 = \cos 2n.$

$\cos 2n[k_1 - k_1\cos 2 + k_2\sin 2]$
$+ \sin 2n[k_2 - k_1\sin 2 - k_2\cos 2] = \cos 2n.$

Comparing L.H.S & R.H.S terms, we have

$k_1 - k_1\cos 2 + k_2\sin 2 = 1.$ ($\because$ co-efficient of $\cos 2n$ is 1)
$k_1(1-\cos 2) + k_2\sin 2 = 1$ — (2)

Similarly $k_2 - k_1\sin 2 - k_2\cos 2 = 0$ ($\because$ co-efficient of $\sin 2n$ is 0)

$k_2(1-\cos 2) - k_1\sin 2 = 0.$

$-k_1\sin 2 + k_2(1-\cos 2) = 0$ — (3)

Solving equation (2) & (3)

$k_1 = \frac{1}{2}$ $\quad k_2 = 0.321$

Now, $y^p(n) = k_1\cos 2n + k_2\sin 2n.$
$= \frac{1}{2}\cos 2n + 0.321\sin 2n.$

$y^f(n) = y^h(n) + y^p(n)$

$y^f(n) = c\,u(n) + \frac{1}{2}\cos 2n + 0.321 \sin 2n.$

Now ~~Let~~ put $n=0$ in above equation.

$y(0) = c\,u(0) + \frac{1}{2}\cos(0) + 0.321 \sin 0$

$y(0) = c + \frac{1}{2} + 0$ $\quad$ ($\because \sin 0 = 0$ & $\cos 0 = 1$)

$y(0) = c + \frac{1}{2}$. — (4)

Put $n=0$ in difference equation.

$y(0) - y(0-1) = x(0)$

$y(0) - y(-1) = \cos 0$

$y(0) - 0 = 1$

$y(0) = 1$,

$\therefore$ Now substitute the value of $y(0)$ in (4).

$\therefore \quad 1 = c + \frac{1}{2}$

$\boxed{c = \frac{1}{2}}$

$\therefore y^f(n) = \frac{1}{2}u(n) + \frac{1}{2}\cos 2n + \frac{}{}0.321 \sin 2n.$

Determine the step response of the system described by difference equation.

$$y(n) + 4y(n-1) + 4y(n-2) = x(n).$$

Given:

$$y(n) + 4y(n-1) + 4y(n-2) = x(n) \quad \text{—(1)}:$$

For step response $x(n) = u(n)$.

The total solution is.

$$y^t(n) = y^n(n) + y^p(n)$$

I) Homogeneous equation,

$y(n) + 4y(n-1) + 4y(n-2) = 0$.

characteristic equation.

$r^2 + 4r + 4 = 0$.

$(r+2)^2 = 0$.

$r_1 = -2$ & $r_2 = -2$.

The homogeneous solution for repeated roots is.

$$y^h(n) = [C_1 + C_2 n.](-2)^n$$

Now for an input $x(n) = u(n)$, the particular solution is of the form,

$$y^p(n) = k\,u(n)$$

Substituting ~~pa~~ above equation in equation (1).

$$k u(n) + 4k u(n-1) + 4k u(n-2) = u(n).$$

For $n=2$, where none of the terms vanish, we get.

$$k u(2) + 4k u(1) + 4k u(0) = u(2)$$

$$k + 4k + 4k = 1.$$

$$9k = 1$$

$$k = 1/9$$

$\therefore\ y^p(n) = \frac{1}{9} u(n)$,

Now $y^t(n) = y^h(n) + y^p(n)$

$$y^t(n) = [C_1 + C_2 n](-2)^n + \frac{1}{9} u(n)$$

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$y^t(n) = [C_1 + C_2 n](-2)^n + 1/9\, u(n)$ for $n \geq 0$. — ②

Now put $n=0$ in above equation.

$y(0) = [C_1 + C_2(0)](-2)^0 + 1/9\,(\phi)$

$y(0) = [C_1 + 0](1) + \phi/9$

$y(0) = C_1 + 1/9$ — ③

Similarly

$y(1) = [C_1 + C_2](-2) + 1/9$

$y(1) = -2C_1 - 2C_2 + 1/9$ — ④

Put $n=0$ in difference equation.

$y(0) + 4y(-1) + 4y(-2) = x(0)$

$y(0) + 0 + 0 = 1 \quad (\because u(0) = 1)$

$\boxed{y(0) = 1}$

Put $n=1$ in difference equation

$y(1) + 4y(0) + 4y(-1) = x(1)$

$y(1) + 4 = 1$

$\boxed{y(1) = -3}$

Substituting the value of $y(0)$ in eq ③.

$1 = C_1 + 1/9$

$C_1 = 1 - 1/9 = 8/9$.

~~similarly~~ substituting the value of $y(1)$ & C_1 in eq ~~③~~ ④

$-3 = -2(8/9) - 2C_2 + 1/9$

$$-3 = -2\left(\frac{8}{9}\right) - 2C_2 + \frac{1}{9}.$$

$$-3 = \frac{-16}{9} + \frac{1}{9} - 2C_2.$$

$$-3 = -\frac{15}{9} - 2C_2.$$

$$+2C_2 = \frac{-15}{9} + 3.$$

$$+2C_2 = \frac{-15+27}{9}$$

$$+2C_2 = \frac{+12}{9}$$

$$\boxed{C_2 = \frac{6}{9} = \frac{2}{3}}$$

Substituting the values of C_1 & C_2 in eq (2)

$$\therefore y(n) = \left(\frac{8}{9} + \frac{2}{3}n\right)(-2)^n u(n) + \frac{1}{9}u(n) \quad n \geq 0.$$

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Find the response of the system described by difference equation:

$$y(n) - \tfrac{1}{2}y(n-1) - \tfrac{1}{2}y(n-2) = \left(\tfrac{1}{2}\right)^n \text{ for } n \geq 0$$

$$y(-1) = 1;\ y(-2) = 0.$$

Sol^n) $y^t(n) = y^h(n) + y^p(n)$ —①.

Given

$$y(n) - \tfrac{1}{2}y(n-1) - \tfrac{1}{2}y(n-2) = \left(\tfrac{1}{2}\right)^n \text{ for } n \geq 0 \quad —②$$

& $y(-1) = 1;\ y(-2) = 0.$

I) Homogeneous equation.

$$y(n) - \tfrac{1}{2}y(n-1) - \tfrac{1}{2}y(n-2) = 0.$$

II) Characteristic equation

$$r^2 - \tfrac{1}{2}r - \tfrac{1}{2} = 0$$

$r_1 = 1$ & $r_2 = -\tfrac{1}{2}.$

$$\therefore y^{(h)}{}_n = c_1 r_1^n + c_2 r_2^n.$$

$$y^{(h)}(n) = c_1(1)^n + c_2\left(-\tfrac{1}{2}\right)^n$$

$$y^{(p)}(n) = k\left(\tfrac{1}{2}\right)^n u(n).$$

Substituting above equation in eq②.

$$k\left(\tfrac{1}{2}\right)^n u(n) - k\tfrac{1}{2}\left(\tfrac{1}{2}\right)^{n-1}u(n-1) - \tfrac{1}{2}\times k\times\left(\tfrac{1}{2}\right)^{n-2}u(n-2) = \left(\tfrac{1}{2}\right)^n.$$

when $n = 2$

$$k\left(\tfrac{1}{2}\right)^2(1) - k\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^1(1) - \tfrac{1}{2}\times k\times\left(\tfrac{1}{2}\right)^0(1) = \left(\tfrac{1}{2}\right)^2.$$

$$\cancel{\tfrac{1}{4}k} - \cancel{\tfrac{1}{4}k} - \tfrac{1}{2}k = \tfrac{1}{4}.$$

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$-\frac{1}{2}k = \frac{1}{4}$

$\boxed{k = -\frac{1}{2}}$

$\therefore y^P(n) = -\frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$

Now $y^t(n) = y^h(n) + y^P(n)$

$$y^t(n) = c_1(1)^n + c_2\left(-\frac{1}{2}\right)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) \quad \text{— (3)}$$

Put $n=0$ in eq (3).

$y(0) = c_1 + c_2 - \frac{1}{2}(1)(1)$

$y(0) = c_1 + c_2 - \frac{1}{2}$ — (4)

Put $n=1$ in eq (3).

$y(1) = c_1 - \frac{c_2}{2} - \frac{1}{4}$ — (5).

Put $n=0$ in eq (2).

$y(0) - \frac{1}{2}y(-1) - \frac{1}{2}y(-2) = 1$

$y(0) - \frac{1}{2}(1) - \frac{1}{2}(0) = 1$ $\left[\because y(-1)=1 \; \& \; y(-2)=0\right]$

$\boxed{y(0) = 1 + \frac{1}{2} = \frac{3}{2}}$

Put $n=1$ in eq (2).

$y(1) - \frac{1}{2}y(0) - \frac{1}{2}y(-1) = \frac{1}{2}$

$y(1) - \frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}(1) = \frac{1}{2}$

$\boxed{y(1) = \frac{1}{2} + \frac{3}{4} + \frac{1}{2} = \frac{7}{4}}$

Substituting the value of $y(0)$ in eq ④.

$$c_1 + c_2 - \frac{1}{2} = \frac{3}{2}$$

$$\boxed{c_1 + c_2 = 2} \quad \text{— ⑥}$$

Substituting the value of $y(1)$ in eq ⑤.

$$c_1 - \frac{c_2}{2} - \frac{1}{4} = \frac{7}{4}.$$

$$c_1 - \frac{c_2}{2} = \frac{8}{4}$$

$$\boxed{c_1 - \frac{c_2}{2} = 2} \quad \text{— ⑦}$$

Solving equations ⑥ & ⑦.

we get,

$$c_1 = 2 \; ; \; c_2 = 0.$$

$$\therefore y^t(n) = 2(1)^n + 0 - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n).$$

$$y^t(n) = 2(1)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n)$$

* Find the solution to the following linear constant co-efficient difference equation $y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = \left\{\frac{1}{4}\right\}^n$ for $n \geq 0$.

with initial conditions $y(-1) = 4$ & $y(-2) = 10$.

Sol^n) Given $y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = \left(\frac{1}{4}\right)^n ; n \geq 0.$

$y(-1) = 4$ & $y(-2) = 10$ — ①

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The total solution

$$y^{(t)}n = y^{n}(n) + y^{p}(n)$$

I) Homogeneous equation

$$y(n) - 3/2\, y(n-1) + 1/2\, y(n-2) = 0$$

II) characteristic equation

$$r^2 - 3/2\, r + 1/2 = 0.$$

$$r_1 = 1 \qquad r_2 = 1/2.$$

$$y^{n}(n) = C_1(1)^n + C_2(1/2)^n.$$

$$y^{p}(n) = K(1/4)^n.$$

Substituting $y^{p}(n)$ in eq (I),

$$K(1/4)^n - 3/2\, K(1/4)^{n-1} + 1/2\, K(1/4)^{n-2} = (1/4)^n.$$

For $n \geq 2$ where none one of the terms vanish.

$$K\left(\frac{1}{4}\right)^2 - 3/2\, K(1/4)\cdot 4 + \frac{K}{2} = \frac{1}{16}.$$

$$\frac{K}{16} - \frac{3K}{8} + \frac{K}{2} = \frac{1}{16}$$

$$K\left[\frac{1}{16} - \frac{3}{8} + 1/2\right] = \frac{1}{16}$$

$$K = 1/3.$$

$$y(t)n = C_1(1)^n + C_2(1/2)^n + 1/3\left(\frac{1}{4}\right)^n.$$

Now;

$$y(0) = C_1 + C_2 + 1/3 \quad -(I)$$

$$\& \quad y(1) = C_1 + \frac{C_2}{2} + \frac{1}{12}. \quad -(II)$$

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ubstituting $n=0$ in difference equation.

$$y(0) - \frac{3}{2}y(-1) + \frac{1}{2}y(-2) = \left(\frac{1}{4}\right)^0$$

$$y(0) - \frac{3}{2}(4) + \frac{1}{2}(10) = 1$$

$$y(0) = 1 + \frac{3}{2}(4) - \frac{10}{2}$$

$$= 1 + 6 - 5$$

$$\boxed{y(0) = 7 - 5 = 2}$$

Ily $y(1) - \frac{3}{2}y(0) + \frac{1}{2}y(-1) = \left(\frac{1}{4}\right)^1$

$$y(1) - \frac{3}{2}(2) + \frac{1}{2}(2) = \frac{1}{4}$$

$$y(1) = \frac{1}{4} + \frac{3}{2}(2) - \frac{1}{2}(4)$$

$$y(1) = \frac{1}{4} + 3 - 2$$

$$\boxed{y(1) = \frac{1}{4} + 1 = \frac{5}{4}}$$

Substituting the value of $y(0)$ & $y(1)$ in ① & ② respectively

$$C_1 + C_2 = 2 - \frac{1}{3}$$

$$C_1 + C_2 = \frac{5}{3} \quad -③$$

Now $C_1 + \frac{C_2}{2} = \frac{5}{4} - \frac{1}{12}$

$$C_1 + \frac{C_2}{2} = \frac{14}{12} \quad -④$$

Solving ③ & ④,

we get,

$$C_1 = \frac{2}{3} \quad C_2 = 1$$

$\therefore\ y^{(t)}n = \frac{2}{3}(1)^n + \left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n.$

$y^{(t)}n = \frac{2}{3}u(n) + \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}\left(\frac{1}{4}\right)^n u(n)$

Extra problems:

i) Find the natural response of the system described by difference equation.

$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1).$

with initial conditions $y(-1) = y(-2) = 1$

Ans: $-3(-1)^n - 2n(-1)^n.$

ii) Find the forced response of the system described by difference equation

$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$

for the input $x(n) = \left(\frac{1}{2}\right)^n u(n).$

Ans: $\frac{2}{3}(-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n.$

iii) Find the response of the system with difference equation:

$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$

for the input $x(n) = \left(\frac{1}{2}\right)^n u(n)$ with initial conditions $y(-1) = y(-2) = 1$

Ans: $-\frac{7}{3}(-1)^n - 2n(-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n$

Determine the step response of the following systems:

i) $y(n) + 3y(n-1) + 2y(n-2) = x(n)$

Ans: $\frac{4}{3}(-2)^n - \frac{1}{2}(-1)^m + \frac{1}{2}$.

ii) $y(n) + 0.4y(n-1) - 0.21y(n-2) = x(n) + 2x(n-1)$

Ans: $-0.5353(-0.7)^n - 0.9857(0.3)^n + 2.521u(n)$

iii) $y(n) - 2\cos\theta\, y(n-1) + y(n-2) = x(n) - \cos\theta\, x(n-1)$

Ans: $\frac{1}{2}\left[1 + \frac{\sin(n+\frac{1}{2})\theta}{\sin\theta/2}\right]$

Find the response of the following difference equations

i) $y(n) + 2y(n-1) + y(n-2) = x(n)$ where $x(n) = \cos 2n$.

ii) $y(n) - 4y(n-1) + 3y(n-2) = x(n)$ for $x(n) = n$

iii) $y(n) - 7y(n-1) + 12y(n-2) = x(n)$ for $x(n) = 2^n u(n)$

[Dif]ferential & Difference Equation Representations [of] LTI system

* Linear constant-coefficient difference & differential equations provide another representation for the input-output characteristics of LTI systems.

* Difference equations are used to represent discrete time systems, while differential equations represent continuous-time systems.

The general form of a linear constant-co-efficient differential equation is

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)}.$$

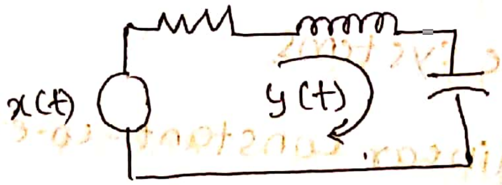
where the a_k & the b_k are constant coefficients of the system
$x(t)$ is the input applied to the system,
and $y(t)$ is the resulting output.

* A linear constant- co-efficient difference equation has a similar form, with the derivatives replaced by delayed values of the input $x[n]$ & output $y[n]$:

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]}. \quad \text{--- (1)}$$

The order of the differential or difference equation is (N, M), representing the number of energy storage devices in the system. Often $N \geq M$, & the order is described using only N.

As an example of a differential equation that describes the behaviour of a physical system, consider the RLC circuit.



Suppose the input is the voltage source $x(t)$ & the output is the current around the loop, $y(t)$.

Then summing the voltage drops around the loop gives

$$R y(t) + L \frac{d}{dt} y(t) + \frac{1}{C} \int_{-\infty}^{t} y(\tau)\, d\tau = x(t)$$

Differentiating both sides w.r.t t.

$$R \frac{d(y(t))}{dt} + L \frac{d^2 y(t)}{dt^2} + \frac{1}{C} y(t) = \frac{d(x(t))}{dt}$$

This differential equation describes the relationship between the current $y(t)$ & the voltage $x(t)$ in the circuit. In this example, the order is $N=2$, & we note that the circuit contains two energy storage devices, a capacitor & an inductor.

Mechanical systems also may be described in terms of differential equations that make use of Newton's law. The behaviour of the MEMS accelerometer ~~[illegible]~~ is given by the differential equation

$$\omega_n^2 y(t) + \frac{\omega_n}{Q}\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) = x(t),$$

where $y(t)$ is the position of the proof mass & $x(t)$ is the external acceleration. This system contains two energy storage mechanisms - a spring & a mass - and the order is again 2.

An example of a second-order difference equation is $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$ — (2)

which may represent the relationship between the input & output signals of a system that process data in a computer. Here, the order is $N=2$, because the difference equation involves $y[n-2]$, implying a maximum memory of 2 in the system output.

- Memory in a descrete-time system is analogous to energy storage in a continuous-time system.

Difference equations are easily rearranged obtain recursive formulas for computing the current output of the system from the input signal & past outputs. We rewrite ① so that y(n) is alone on the left-hand side

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y(n-k)$$

This equation indicates how to obtain y[n] from the present & past values of the input & the past values of the o/p. such equations are often used to implement discrete time systems in a computer ~~DTS in a~~ discrete time systems in a computer from x(n).

Consider computing y[n] for $n \geq 0$ for eq ②,

$$y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]$$

Now

$$y(0) = x(0) + 2x(-1) - y(-1) - \frac{1}{4}y(-2)$$

$$y[1] = x(1) + 2x[0] + y(0) - \frac{1}{4}y(-1)$$

$$y(2) = x[2) + 2x[1] + y[1] - \frac{1}{4}(0)$$

In each equation, the current o/p is computed from the i/p & past values of the output. In order to begin this process at time n=0, we must know the two most recent past values of the output, namely, y[-1] & y[-2]. these values are known as <u>initial conditions</u>.

The initial conditions summarize all the information about the system's past that is needed to determine future outputs.

- In general, the number of initial conditions required to determine the output is equal to the maximum memory of the system.

It is common to choose $n=0$ or $t=0$ as the starting time for solving a difference or differential equation, respectively. In this case, the initial conditions for an Nth-order difference equation are the N values

$y[-N], y[-N+1], \ldots, y[-1]$,

and the initial conditions for an Nth-order differential equation are the values of the first N derivatives of the output -

that is,

$$y[t]\Big|_{t=0^-}, \frac{d}{dt}y(t)\Big|_{t=0^-}, \frac{d^2}{dt^2}y(t)\Big|_{t=0^-}, \ldots, \frac{d^{N-1}}{dt^{N-1}}y(t)\Big|_{t=0^-}$$

The initial conditions in a differential-equation description of an LTI system are directly related to the initial values of the energy storage devices in the system, such as initial voltages on capacitors & initial currents through inductors. As in the discrete-time case, the initial conditions summarize all information about the past history of the system that can affect future outputs. Hence, initial conditions also represent the "memory" of continuous-time systems.